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# Dynamics of ferromagnetic spin glass: randomly canted ferromagnet versus skewed spin glass

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**Abstract**

A ferromagnetic spin glass (FSG) is one of the three isotropic and homogeneous phases of the long-range partially ordered magnets with spin and atomic disorder which are selected by symmetry (Andreev 1978 *Sov. Phys.—JETP* **47** 411) (the others are genuine and antiferromagnetic spin glasses). The linear dynamical response to a magnetic field of two sub-phases of a FSG with drastically different dynamics, a randomly canted ferromagnet, in which the component spins create an acute angle with the summary magnetic moment, and a less-ordered skewed spin glass is analysed in the spin-wave approximation in the framework of phenomenological theory. The spin-wave damping coefficients and frequency shifts due to a magnon–magnon interaction are evaluated as functions of temperature and wavevector as well as the spectral-weight functions of the linear response to a magnetic field and the neutron scattering cross section which provides the possibility for experimental verification of the results. Substantial differences in the spin-wave characteristics of the FSG compared to those of the Heisenberg spin glass and the Heisenberg ferromagnet are found to be due to non-linear anisotropy effects in a FSG.

## 1. Introduction

Ferromagnetic spin glasses (FSGs) have been extensively studied during the last three decades. Spontaneously magnetized phases are present in a large number of magnetic materials with random distribution of magnetic ions. There are metallic (e.g. $Ni_{1-x}Mn_x$) [1], semiconducting (e.g. (III, Mn)V compounds) [2, 3] and insulating FSGs (e.g. $Eu_xSr_{1-x}S$) [4]. Therefore, the exchange interactions responsible for the FSG ordering are different (RKKY, superexchange, etc). We present a general quantum description of the dynamics of a FSG applicable also to perfectly ordered non-collinear ferromagnets in the framework of the spin-wave approximation, which is one of the well known problems of the theory of magnetism. Recently the problem

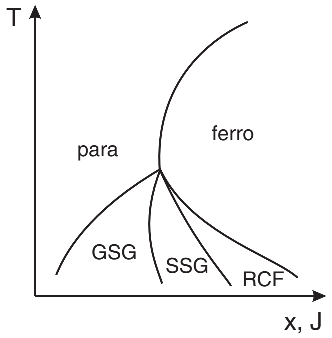


**Figure 1.** Schematic temperature–mean exchange energy $J$ (or, equivalently, temperature–concentration of ferromagnetic ions $x$) phase diagram of the exchange magnet with competing ferro- and antiferromagnetic interactions.

became interesting not just from the purely academic point of view since potentially important new applications of diluted magnets are being considered.

Since microscopic models of disorder are not general, the macroscopic description of dynamics is especially valid for the disordered magnets. The theory of spin-wave dynamics of the isotropic homogeneous phases of spin glasses was developed since the pioneering work [5] by Halperin and Saslow via the work [6] by Andreev who classified the disordered phases and included non-linearity in the description. According to his group theory analysis, a FSG is one of the three isotropic disordered phases selected by symmetry (next to a genuine spin glass (GSG) and an antiferromagnetic spin glass) [6, 7]. Therefore it is a fundamental phase for theoretical considerations. FSGs can be divided into: randomly canted ferromagnets (RCF), in which the local spin components create an acute angle with the global magnetic moment, and skewed spin glasses (SSG), in which the local spin components create any angle with $\boldsymbol{M}$, (the nomenclature was introduced in [8]). We study the dynamics of RCF separately from the dynamics of SSG because they differ in terms of the number of dynamical degrees of freedom. According to [7], the description for RCF is also applicable for non-collinear ferromagnets whereas the description for SSG is applicable for non-collinear ferrimagnets.

More strict characterization of both FSG sub-phases demands the introduction of order parameters. They are: the magnetization vector $\boldsymbol{M} = \gamma/V\langle\sum_{\mu=1}^{N} \boldsymbol{s}_\mu\rangle$, where $\gamma$ denotes a gyromagnetic ratio and the bracket denotes the thermal average, and the parameters of Edwards–Anderson type $q_i = \gamma^2/V \sum_{\mu=1}^{N}\langle s_\mu^i\rangle^2$. The equivalency to the diagonal components of the static equilibrium susceptibility $\chi_{ii}$ is discussed in [9]. With $\boldsymbol{M} = M\hat{z}$, RCF is characterized by $\chi_{zz} = \chi_\parallel \neq 0$, $\chi_{xx} = \chi_{yy} = 0$ ($q_z = M^2V/N$), unlike SSG for which $\chi_{xx} = \chi_{yy} = \chi_\perp \neq 0$ ($q_z \neq M^2V/N$). There is a general scheme of the phase diagram of the exchange magnets with competing ferromagnetic and antiferromagnetic interactions arising from different models [10, 11], containing RCF and SSG, as shown in figure 1.

I develop the method proposed in [12] for the description of GSG dynamics, carrying out the canonical quantization of spin waves and evaluating the linear response to a magnetic field using the Green function technique. Studying the poles of the elements of the diagonalized magnetic susceptibility tensor, temperature-dependent non-linear corrections to the spin-wave frequencies and their coefficients of damping due to the magnon–magnon interaction are estimated. These considerations lead to the conclusion that differences in the relaxation of magnons compared to the relaxation in the Heisenberg ferromagnet (see also [13]) result from the presence of anisotropy in a FSG which is essential for stabilization of the non-collinear structure. The special goal of the work is a comparison between the spin-wave relaxation characteristics (damping coefficients) for RCF and SSG.

This paper is organized as follows. In section 2, we present the Andreev spin-wave Lagrangian for a FSG, analysing the consequences of its symmetry in detail. In section 3, the quantum spin-wave theory for RCF is developed and section 4 is devoted to the quantization of spin waves relevant to SSG. The summary of the results and the outlook for additional information on the macroscopic description of the dynamics of a FSG is presented in section 5.

## 2. Spin-wave Lagrangian

The crucial point in writing the macroscopic Lagrangian of spin waves is the determination of its external and internal symmetry. The external symmetry of the Lagrangian of a FSG is $SO(3) \times P$, where $P$ denotes the space inversion, whereas the internal symmetry follows from the symmetry of spin interactions. Following Andreev we assume the presence of isotropic two-ion exchange interactions and anisotropic one-ion interactions described with the microscopic Hamiltonian

$$\mathcal{H} = -\tfrac{1}{2} \sum_{\mu,\nu} J_{\mu\nu} \boldsymbol{s}_\mu \cdot \boldsymbol{s}_\nu - \sum_{\mu} \boldsymbol{H}_\mu \cdot \boldsymbol{s}_\mu, \tag{1}$$

where the exchange constants $J_{\mu\nu}$ change their sign with the distance between ions and the effective anisotropy field $\boldsymbol{H}_\mu$ acting on the spin of the $\mu$th site is randomly distributed. The possible presence of different magnetic ions in the system can be included via averaging the gyromagnetic ratio. Some parameters of spin waves (energy gap due to magnon–magnon interactions) can be identified, however, as so-called exchange-anisotropy parameters, which is not in contradiction with the model. The origin of this effect in the exchange magnets with one-ion anisotropy has been clarified in [14]. Its presence may be the result of linearization of the microscopic spin Hamiltonian since the canonical transformation which eliminates linear terms from the Hamiltonian modifies the bilinear two-ion interaction term. However, it does not change the symmetry of the Hamiltonian as a whole. As a consequence, even in the case of a lack of the spin-wave gap due to the bilinear anisotropy terms of the Hamiltonian, it may appear due to non-linearity and it may be found from macroscopic considerations if the corresponding Lagrangian is not Lorentz-invariant.

The spin-wave Lagrangian consists of a kinetic part (depending on time derivatives of the dynamical parameter), a gradient part (depending on space derivatives) and a non-differential part. The kinetic part is determined by the densities of the components of the dynamically induced macroscopic magnetic moment, see below; the gradient part of the Lagrangian describes two-ion exchange interactions, so it must be invariant under the isotopic space rotations. The non-differential part describes the one-ion anisotropy and breaks the internal $SO(3)$ symmetry. It is possible to introduce a gradient part breaking the internal $SO(3)$ symmetry, as is done in [15], in order to include the two-ion anisotropy effects of the physical nature. However, we neglect them as weak effects.

The dynamical parameter is the parameter of the spin (isotopic) space rotation. We use the so-called vector parametrization of the $SO(3)$ group, as was done in [6, 7]. The parameter $\boldsymbol{\varphi}(\mathbf{x}, t) = \boldsymbol{n} \tan(\theta/2)$ is related to the rotation through the angle $\theta$, $(0 < \theta < \pi/2)$, about the vector $\boldsymbol{n}$, $(|\boldsymbol{n}| = 1)$. The transformation-matrix elements are of the form

$$O_{ij}(\boldsymbol{\varphi}) = \delta_{ij} + 2(\varphi_i \varphi_j - \varphi^2 \delta_{ij} + \epsilon_{ikj} \varphi_k)/(1 + \varphi^2). \tag{2}$$

We use the Einstein summation convention.

The Lagrangian is a combination of invariants of the point symmetry group, built of vector and tensor quantities. To ensure the internal $SO(3)$ symmetry of the differential part of the Lagrange function, the differential invariants are built of vectors and tensors under those transformations: the spontaneous-magnetic-moment density $\boldsymbol{M}(\varphi) = \hat{O}(\varphi)\boldsymbol{M}_{\rm o}$, where $\boldsymbol{M}_{\rm o}$ is the equilibrium moment, and right differential forms $\dot{\boldsymbol{\Omega}} \equiv \delta\boldsymbol{\Omega}/\partial t$, $\boldsymbol{\Omega}_{,m} \equiv \delta\boldsymbol{\Omega}/\partial x_m$ connected to the right Cartan forms via the relations

$$\dot{O}_{kj}(O^{-1})_{ji} = -2\varepsilon_{kli}\dot{\Omega}_l = -2\varepsilon_{kli}\frac{\dot{\varphi}_l + (\varphi \times \dot{\varphi})_l}{1+\varphi^2}, \tag{3a}$$

$$O_{kj,m}(O^{-1})_{ji} = -2\varepsilon_{kli}\Omega_{l,m} = -2\varepsilon_{kli}\frac{\varphi_{l,m} + (\varphi \times \varphi_{,m})_l}{1+\varphi^2} \tag{3b}$$

invariant under the right transformations $\hat{O}(\varphi) \to \hat{O}(\varphi)\hat{O}(\varphi')$ (some of the cited authors use equivalent invariants built of scalar left forms) [15]. The non-differential anisotropy invariants are built of the vector $\boldsymbol{M}(\varphi)$ and of the matrix $\hat{O}(\varphi)$, which is a tensor under space rotations but not under spin rotations.

The transformation of the dynamical parameter connected to the spin-space rotation

$$\varphi \to \varphi' = (\epsilon + \varphi + \epsilon \times \varphi)/(1 - \epsilon \cdot \varphi) \tag{4}$$

follows from the rotation-group multiplying rule $\hat{O}(\varphi') = \hat{O}(\epsilon)\hat{O}(\varphi)$. The linear part of this transformation determines the densities $m_i = (\partial\mathcal{L}/\partial\dot{\varphi}_j)(\partial\delta\varphi_j/\partial\epsilon_i)$ of the components of the magnetic moment (see above). They are postulated to be composed of spontaneous, dynamically induced, and external field parts as

$$\boldsymbol{m} = \boldsymbol{M} + \frac{2\chi_\perp}{\gamma}\dot{\boldsymbol{\Omega}} + \frac{2(\chi_\parallel - \chi_\perp)}{\gamma M^2}(\boldsymbol{M}\cdot\dot{\boldsymbol{\Omega}})\boldsymbol{M} + \hat{\chi}\boldsymbol{H}. \tag{5}$$

Here $\chi_{ij} = \chi_\perp\delta_{ij} + (\chi_\parallel - \chi_\perp)M_{{\rm o}i}M_{{\rm o}j}/M^2$ is the equilibrium static susceptibility and $\boldsymbol{H}$ denotes the external magnetic field.

The Lagrange function for a FSG is of the form

$$\mathcal{L} = a\dot{\boldsymbol{\Omega}}\cdot\dot{\boldsymbol{\Omega}} + a'(\boldsymbol{M}\cdot\dot{\boldsymbol{\Omega}})^2 + a''\boldsymbol{M}\cdot\dot{\boldsymbol{\Omega}} + b(\boldsymbol{M}\cdot\boldsymbol{\Omega}_{,i})^2 + b'\boldsymbol{\Omega}_{,i}\cdot\boldsymbol{\Omega}_{,i}$$
$$+ \frac{2a}{a''}\boldsymbol{H}\cdot\dot{\boldsymbol{\Omega}} + \boldsymbol{M}\cdot\boldsymbol{H} + \frac{2a'}{a''}\boldsymbol{M}\cdot\boldsymbol{H}(\boldsymbol{M}\cdot\dot{\boldsymbol{\Omega}}) - U_{\rm an}, \tag{6}$$

where $a = 2\chi_\perp/\gamma^2$; $a' = 2(\chi_\parallel - \chi_\perp)/\gamma^2M^2$; $a'' = 2/\gamma$; $b = -2(\chi_\parallel c_\parallel^2 - a_\perp\gamma M)/\gamma^2M^2$ and $b' = -2a_\perp M/\gamma$ [6]. Here the anisotropy energy

$$U_{\rm an} = [\alpha_1\varphi^2 + \alpha_2/M^2(M_{\rm o}\times\varphi)^2 + \alpha_3\varphi^4 + \alpha_4/M^2\varphi^2(M_{\rm o}\times\varphi)^2](1+\varphi^2)^{-2} \tag{7}$$

is a combination of the non-differential invariants $O_{ii}$, $O_{ij}O_{ji}$, $M_iO_{ij}M_j$, $M_iO_{ij}O_{jk}M_k$, $M_iO_{ij}M_jO_{kk}$.

Let us notice that, since the static part of the thermodynamical potential of a FSG contains the only bilinear exchange invariant, $\boldsymbol{M}\cdot\boldsymbol{M}$. The variation of this term is of the form $\delta(\boldsymbol{M}\cdot\boldsymbol{M}) = 2\boldsymbol{M}\cdot\delta\boldsymbol{M} + \delta\boldsymbol{M}\cdot\delta\boldsymbol{M}$, the exchange part of the spin-wave Lagrangian is proportional to the invariant $\boldsymbol{M}_{,i}\cdot\boldsymbol{M}_{,i} = M^2(\boldsymbol{\Omega}_{,i}\cdot\boldsymbol{\Omega}_{,i}) - (\boldsymbol{M}\cdot\boldsymbol{\Omega}_{,i})^2$, different from the gradient part of (6) for $b'M^2 \neq -b$, [16]. I interpret this fact as the influence of the one-ion anisotropy as discussed at the beginning of this section. Let us notice also that, according to the considerations of [16], one cannot apply a similar analysis to the GSG Lagrangian since there are no multipole moments for GSG (order parameters being linear in spin components) [6].

For RCF $\chi_\perp = 0$, which means that the first term of the Lagrangian vanishes. In this case two equations of motion are dependent on each other and the number of dynamical degrees of freedom reduces. The solutions of the linearized equations of motion describe two spin-wave modes, one shear ferromagnetic-like mode and one longitudinal mode with related quadratic and linear dispersions:

$$\omega_{\perp\mathbf{k}} = \frac{-b'k^2 + \alpha + 2MH}{\Theta} \simeq a_\perp k^2 + \gamma H + \frac{\gamma(\alpha_1 + \alpha_2)}{2M}, \tag{8}$$

$$\omega_{\|\mathbf{k}}^2 = c_\|^2 k^2 + \frac{\alpha_1 \gamma^2}{2\chi_\|}, \tag{9}$$

where $\Theta \equiv 2[M + (\chi_\| - \chi_\perp)H]/\gamma$, $a_\perp$ denotes a shear-mode parameter of dispersion and $c_\|$ denotes a velocity of the longitudinal spin waves.

For the SSG phase ($\chi_\perp \neq 0$), there are three spin-wave modes: two shear ferrimagnetic-like modes with related dispersions:

$$\omega_{\mathbf{k}\pm} = \frac{\gamma}{2}\left\{\pm\left(\frac{M}{\chi_\perp} - (1+\xi)H\right) + \left[\left(\frac{M}{\chi_\perp} + (1-\xi)H\right)^2 + 4\xi H^2 + \frac{4M}{\gamma\chi_\perp}a_\perp k^2 + \frac{2(\alpha_1+\alpha_2)}{\chi_\perp}\right]^{1/2}\right\}, \tag{10}$$

where $\xi = 1 - \chi_\|/\chi_\perp$, and one longitudinal mode of the dispersion (9), according to the results of [8, 17].

Below we will quantize the spin waves of RCF and SSG and analyse their dynamics separately.

## 3. Dynamics of randomly canted ferromagnet

### *3.1. Quantization of Hamiltonian and magnetization*

The quantization of elementary excitations of the RCF is a problem in itself, as in the case of the so-called 'light-cone quantization formalism' for elementary particles (see, e.g. [18]), because of the constrained character of its dynamics. I write the Hamiltonian using the canonical momenta of the form

$$\pi_i = \partial\mathcal{L}/\partial\dot{\varphi}_i = \left[2a'\boldsymbol{M}\cdot\dot{\boldsymbol{\Omega}} + a'' + \frac{2a'MH}{a''}\left(1 - 2\frac{\varphi_x^2+\varphi_y^2}{1+\varphi^2}\right)\right]\mu_i, \tag{11}$$

where $\boldsymbol{\mu} \equiv \frac{\boldsymbol{M}_o + (\boldsymbol{\varphi}\times\boldsymbol{M}_o)}{1+\varphi^2}$. It is independent of two of them:

$$\mathcal{H} = \frac{1}{4a'}\left[\frac{\pi_z}{\mu_z} - a'' - \frac{2a'MH}{a''}\left(1 - 2\frac{\varphi_x^2+\varphi_y^2}{1+\varphi^2}\right)\right]^2 - b(\boldsymbol{M}\cdot\boldsymbol{\Omega}_i)^2 - b'\boldsymbol{\Omega}_i\cdot\boldsymbol{\Omega}_i - MH\left(1 - 2\frac{\varphi_x^2+\varphi_y^2}{1+\varphi^2}\right) + U_{\mathrm{an}}. \tag{12}$$

The form of expression (11) leads to the vector equation

$$\boldsymbol{\pi}\times\boldsymbol{\mu} = 0 \tag{13}$$

that contains two scalar equations defining two second-kind constraints, [19] (reducing two variables of the phase space) and one identity. Let us define new variables:

$$\begin{aligned} Q_1 &= \varphi_z + \frac{\varphi_x\pi_x}{\pi_z}, & \Pi_1 &= \pi_z, \\ Q_2 &= \frac{1}{2^{1/2}\Theta}(\pi_y - \varphi_x\pi_z), & \Pi_2 &= -\frac{\Theta}{2^{1/2}}\left(\frac{\pi_x}{\pi_z} + \varphi_y\right), \\ Q_3 &= \pi_x - \varphi_y\pi_z \approx 0, & \Pi_3 &= -\frac{1}{2}\left(\frac{\pi_y}{\pi_z} + \varphi_x\right) \approx 0, \end{aligned} \tag{14}$$

satisfying canonical Poisson relations, where $\Theta = 2M/\gamma$ and the standard notation for the constraint equations is used [19], and write the Hamiltonian in them as

$$\begin{aligned}\mathcal{H} = \frac{1}{4a'M^2}\bigg[\Pi_1(1+\mathcal{O}) - \Theta + \frac{2a'M^2H}{a''(1+\mathcal{O})}\bigg(\Theta^2\frac{Q_2^2}{\Pi_1^2} + \frac{1}{\Theta^2}\Pi_2^2\bigg)\bigg]^2 \\ - \frac{M^2b}{(1+\mathcal{O})^2}\bigg(Q_{1,i} - \frac{Q_2\Pi_{2,i}}{\Pi_1}\bigg)^2 - \frac{b'}{(1+\mathcal{O})^2}\bigg[\frac{\Theta^2}{2}\bigg(\frac{Q_{2,i}}{\Pi_1} - \frac{\Pi_{1,i}Q_2}{\Pi_1^2}\bigg)^2 \\ + \frac{1}{2\Theta^2}\Pi_{2,i}^2 + \bigg(Q_{1,i} + \frac{1}{2}\frac{\Pi_{1,i}Q_2\Pi_2}{\Pi_1^2} - \frac{1}{2}\frac{Q_{2,i}\Pi_2 + Q_2\Pi_{2,i}}{\Pi_1}\bigg)^2 \\ + \frac{1}{2\Theta^2}\bigg(-Q_1\Pi_{2,i} + Q_{1,i}\Pi_2 - \frac{1}{2}\frac{Q_{2,i}\Pi_2^2}{\Pi_1} + \frac{1}{2}\frac{\Pi_{1,i}Q_2\Pi_2^2}{\Pi_1^2}\bigg)^2 \\ + \frac{\Theta^2}{2}\bigg(-\frac{Q_1^{,i}Q_2}{\Pi_1} + \frac{1}{2}\frac{Q_2^2\Pi_{2,i}}{\Pi_1^2} + \frac{Q_1Q_{2,i}}{\Pi_1} - \frac{Q_1\Pi_{1,i}Q_2}{\Pi_1^2}\bigg)^2 \\ + \frac{1}{4}\bigg(\frac{Q_{2,i}\Pi_2}{\Pi_1} - \frac{\Pi_{1,i}Q_2\Pi_2}{\Pi_1^2} - \frac{Q_2\Pi_{2,i}}{\Pi_1}\bigg)^2\bigg] \\ - MH\bigg[1 - \bigg(\Theta^2\frac{Q_2^2}{\Pi_1^2} + \frac{1}{\Theta^2}\Pi_2^2\bigg)(1+\mathcal{O})^{-1}\bigg] \\ + \bigg[\alpha_1\mathcal{O} + \alpha_3\mathcal{O}^2 + \alpha_2\bigg(\frac{\Theta^2}{2}\frac{Q_2^2}{\Pi_1^2} + \frac{1}{2\Theta^2}\Pi_2^2\bigg) \\ + \alpha_4\mathcal{O}\bigg(\frac{\Theta^2}{2}\frac{Q_2^2}{\Pi_1^2} + \frac{1}{2\Theta^2}\Pi_2^2\bigg)^2\bigg](1+\mathcal{O})^{-2}, \qquad (15)\end{aligned}$$

$$\mathcal{O} = \frac{\Theta^2}{2}\frac{Q_2^2}{\Pi_1^2} + \frac{1}{2\Theta^2}\Pi_2^2 + Q_1^2 - \frac{Q_1Q_2\Pi_2}{\Pi_1} + \frac{1}{4}\frac{Q_2^2\Pi_2^2}{\Pi_1^2}. \qquad (16)$$

The additional canonical transformation $\Pi_1' = \Pi_1 - \Theta$ eliminates linear terms from the Hamiltonian.

The canonically quantized boson fields and momenta:

$$\begin{aligned}Q_1 &= \frac{\hbar^{1/2}}{2(Va'M^2)^{1/2}}\sum_{\mathbf{k}}\omega_{\|\mathbf{k}}^{-1/2}(b_{\mathbf{k}} + b_{-\mathbf{k}}^{\dagger})\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot x}, \\ \Pi_1' &= \mathrm{i}\frac{(\hbar a'M^2)^{1/2}}{V^{1/2}}\sum_{\mathbf{k}}\omega_{\|\mathbf{k}}^{1/2}(b_{\mathbf{k}}^{\dagger} - b_{-\mathbf{k}})\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot x}, \\ Q_2 &= \frac{\hbar^{1/2}}{(2V\Theta)^{1/2}}\sum_{\mathbf{k}}(a_{\mathbf{k}} + a_{-\mathbf{k}}^{\dagger})\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot x}, \\ \Pi_2 &= \mathrm{i}\frac{(\hbar\Theta)^{1/2}}{(2V)^{1/2}}\sum_{\mathbf{k}}(a_{\mathbf{k}}^{\dagger} - a_{-\mathbf{k}})\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot x}\end{aligned} \qquad (17)$$

(where $a_{\mathbf{k}}^{(\dagger)}$ and $b_{\mathbf{k}}^{(\dagger)}$ denote annihilation (creation) operators for the shear and longitudinal modes, respectively) diagonalize the bilinear part of the Hamiltonian $\mathcal{H}_2$:

$$\mathcal{H}_2 = \hbar\sum_{\mathbf{k}}\omega_{\|\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \hbar\sum_{\mathbf{k}}\omega_{\perp\mathbf{k}}a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}}. \qquad (18)$$

We consider the quantized Hamiltonian up to the fourth order of expansion in the one-particle operators:

$$\mathcal{H} \simeq \mathcal{H}_2 + \mathcal{H}_3 + \mathcal{H}_4, \qquad (19)$$

treating the bilinear term as the main term, $\mathcal{H}_o = \mathcal{H}_2$, and the next two terms as a perturbation $\mathcal{H}_{int} = \mathcal{H}_3 + \mathcal{H}_4$. Upon quantization, the third-order term is transformed into

$$\begin{aligned}
\mathcal{H}_3 = \mathcal{H}_{3A} + \mathcal{H}_{3B} = \sum_{\mathbf{123}} [&A_{\rm I}(\mathbf{123})(\mathrm{i}b_{\mathbf{1}}^\dagger a_{\mathbf{2}} a_{\mathbf{3}} + \text{h.c.})\Delta(\mathbf{1}-\mathbf{2}-\mathbf{3}) \\
&+ A_{\rm II}(\mathbf{123})(\mathrm{i}b_{\mathbf{1}}^\dagger a_{\mathbf{2}}^\dagger a_{\mathbf{3}} + \text{h.c.})\Delta(\mathbf{1}+\mathbf{2}-\mathbf{3}) \\
&+ A_{\rm III}(\mathbf{123})(\mathrm{i}b_{\mathbf{1}}^\dagger a_{\mathbf{2}}^\dagger a_{\mathbf{3}}^\dagger + \text{h.c.})\Delta(\mathbf{1}+\mathbf{2}+\mathbf{3}) \\
&+ A_{\rm IV}(\mathbf{123})(\mathrm{i}b_{\mathbf{1}}^\dagger a_{\mathbf{2}} a_{\mathbf{3}}^\dagger + \text{h.c.})\Delta(\mathbf{1}-\mathbf{2}+\mathbf{3})] \\
&+ \sum_{\mathbf{123}} [B_{\rm I}(\mathbf{123})(\mathrm{i}b_{\mathbf{1}}^\dagger b_{\mathbf{2}} b_{\mathbf{3}} + \text{h.c.})\Delta(\mathbf{1}-\mathbf{2}-\mathbf{3}) \\
&- B_{\rm I}(\mathbf{312})(\mathrm{i}b_{\mathbf{1}}^\dagger b_{\mathbf{2}}^\dagger b_{\mathbf{3}} + \text{h.c.})\Delta(\mathbf{1}+\mathbf{2}-\mathbf{3}) \\
&+ B_{\rm II}(\mathbf{123})(\mathrm{i}b_{\mathbf{1}}^\dagger b_{\mathbf{2}}^\dagger b_{\mathbf{3}}^\dagger + \text{h.c.})\Delta(\mathbf{1}+\mathbf{2}+\mathbf{3}) \\
&- B_{\rm I}(\mathbf{213})(\mathrm{i}b_{\mathbf{1}}^\dagger b_{\mathbf{2}} b_{\mathbf{3}}^\dagger + \text{h.c.})\Delta(\mathbf{1}-\mathbf{2}+\mathbf{3})]
\end{aligned} \tag{20}$$

where the scattering amplitudes are of the form

$$\begin{aligned}
&A_{\rm I}(\mathbf{123}) = \mathcal{C}_A(\omega_{\|\mathbf{1}}^{3/2} - 2\omega_{\perp\mathbf{2}}\omega_{\|\mathbf{1}}^{1/2}), \\
&A_{\rm II}(\mathbf{123}) = \mathcal{C}_A\left[\left(\frac{\gamma M}{\chi_\|} + 2\gamma H\right)\omega_{\|\mathbf{1}}^{1/2} + \left(-c_\|^2 + \frac{\gamma M}{\chi_\|}a_\perp\right)\frac{|\mathbf{2}|^2 - |\mathbf{3}|^2}{\omega_{\|\mathbf{1}}^{1/2}} - 2\omega_{\perp\mathbf{2}}\omega_{\|\mathbf{1}}^{1/2}\right], \\
&A_{\rm III}(\mathbf{123}) = \mathcal{C}_A(-\omega_{\|\mathbf{1}}^{3/2} - 2\omega_{\perp\mathbf{2}}\omega_{\|\mathbf{1}}^{1/2}), \\
&A_{\rm IV}(\mathbf{123}) = \mathcal{C}_A\left[\left(\frac{\gamma M}{\chi_\|} + 2\gamma H\right)\omega_{\|\mathbf{1}}^{1/2} + \left(-c_\|^2 + \frac{\gamma M}{\chi_\|}a_\perp\right)\frac{|\mathbf{3}|^2 - |\mathbf{2}|^2}{\omega_{\|\mathbf{1}}^{1/2}} - 2\omega_{\perp\mathbf{2}}\omega_{\|\mathbf{1}}^{1/2}\right], \\
&B_{\rm I}(\mathbf{123}) = \mathcal{C}_B\frac{\omega_{\|\mathbf{1}} - \omega_{\|\mathbf{2}} - \omega_{\|\mathbf{3}}}{(\omega_{\|\mathbf{1}}\omega_{\|\mathbf{2}}\omega_{\|\mathbf{3}})^{1/2}}, \qquad B_{\rm II}(\mathbf{123}) = \mathcal{C}_B\frac{\omega_{\|\mathbf{1}} + \omega_{\|\mathbf{2}} + \omega_{\|\mathbf{3}}}{(\omega_{\|\mathbf{1}}\omega_{\|\mathbf{2}}\omega_{\|\mathbf{3}})^{1/2}},
\end{aligned} \tag{21}$$

with $\mathcal{C}_A = \frac{\hbar^{3/2}}{V^{1/2}}\frac{\gamma}{2^{5/2}\chi_\|^{1/2}}\frac{\chi_\|}{\gamma M}$, $\mathcal{C}_B = \frac{\hbar^{3/2}}{V^{1/2}}\frac{\gamma}{24\cdot 2^{1/2}\chi_\|^{1/2}}$. In order to simplify writing the expressions, we denote the wavevectors $\mathbf{k}_1, \mathbf{k}_2, \ldots$ by $\mathbf{1}, \mathbf{2}\ldots$ and we do not use commas between the arguments of amplitudes. The symbol $\Delta(\ldots)$ denotes the Kronecker $\delta$ expressing the conservation of momentum ($\Delta(\mathbf{k}_1 - \mathbf{k}_2) \equiv \delta_{\mathbf{k}_1,\mathbf{k}_2}$).

Since the fourth-order term becomes complex, we will take into account only some fourth-order interaction processes. The amplitudes of these are of the lowest order in $T/\frac{\hbar\gamma M}{\chi_\|}$ when assuming that the magnons contributing to them have energies of the order of $T$ (valid for further estimations) and they conserve the number of magnons. We write the corresponding part of the Hamiltonian $\mathcal{H}_4$ as

$$\begin{aligned}
\mathcal{H}_4 = \mathcal{H}_{4C} + \mathcal{H}_{4D} + \mathcal{H}_{4E} = \sum_{\mathbf{1234}} &C_{\rm I}(\mathbf{1234})[(a_{\mathbf{1}}^\dagger a_{\mathbf{2}}^\dagger a_{\mathbf{3}} a_{\mathbf{4}} + \text{h.c.})\Delta(\mathbf{1}+\mathbf{2}-\mathbf{3}-\mathbf{4}) \\
&+ (a_{\mathbf{1}}^\dagger a_{\mathbf{2}} a_{\mathbf{3}}^\dagger a_{\mathbf{4}} + \text{h.c.})\Delta(\mathbf{1}-\mathbf{2}+\mathbf{3}-\mathbf{4})] \\
&+ \sum_{\mathbf{1234}} C_{\rm II}(\mathbf{1234})(a_{\mathbf{1}}^\dagger a_{\mathbf{2}} a_{\mathbf{3}} a_{\mathbf{4}}^\dagger + \text{h.c.})\Delta(\mathbf{1}-\mathbf{2}+\mathbf{3}-\mathbf{4}) \\
&+ \sum_{\mathbf{1234}} D_{\rm I}(\mathbf{1234})[(b_{\mathbf{1}}^\dagger b_{\mathbf{2}}^\dagger b_{\mathbf{3}} b_{\mathbf{4}} + \text{h.c.})\Delta(\mathbf{1}+\mathbf{2}-\mathbf{3}-\mathbf{4}) \\
&+ (b_{\mathbf{1}}^\dagger b_{\mathbf{2}} b_{\mathbf{3}}^\dagger b_{\mathbf{4}} + \text{h.c.})\Delta(\mathbf{1}-\mathbf{2}-\mathbf{3}+\mathbf{4})] \\
&+ \sum_{\mathbf{1234}} D_{\rm II}(\mathbf{1234})(b_{\mathbf{1}}^\dagger b_{\mathbf{2}} b_{\mathbf{3}} b_{\mathbf{4}}^\dagger + \text{h.c.})\Delta(\mathbf{1}-\mathbf{2}+\mathbf{3}-\mathbf{4})
\end{aligned}$$

$$+\sum_{\mathbf{1234}} E(\mathbf{1234})[(b_{\mathbf{1}}^{\dagger}b_{\mathbf{2}}a_{\mathbf{3}}a_{\mathbf{4}}^{\dagger}+\text{h.c.})\Delta(\mathbf{1}-\mathbf{2}-\mathbf{3}+\mathbf{4})$$
$$+(b_{\mathbf{1}}^{\dagger}b_{\mathbf{2}}a_{\mathbf{3}}^{\dagger}a_{\mathbf{4}}+\text{h.c.})\Delta(\mathbf{1}-\mathbf{2}+\mathbf{3}-\mathbf{4})], \quad (22)$$

where the amplitudes are the following:

$$C_{\mathrm{I}}(\mathbf{1234}) = \frac{\hbar^2}{V}\left[\frac{\gamma^2}{32\chi_\parallel} - \frac{\gamma a_\perp}{8M}(\mathbf{1}\cdot\mathbf{4}+\mathbf{2}\cdot\mathbf{3}) + \frac{\chi_\parallel c_\parallel^2}{16M^2}(\mathbf{1}\cdot\mathbf{4}+\mathbf{2}\cdot\mathbf{3})\right],$$
$$C_{\mathrm{II}}(\mathbf{1234}) = \frac{\hbar^2}{V}\frac{\chi_\parallel c_\parallel^2}{16M^2}(\mathbf{1}\cdot\mathbf{4}+\mathbf{2}\cdot\mathbf{3}),$$
$$D_{\mathrm{I}}(\mathbf{1234}) = \frac{\hbar^2}{V}\frac{\gamma^2}{8\chi_\parallel(\omega_{\parallel\mathbf{1}}\omega_{\parallel\mathbf{2}}\omega_{\parallel\mathbf{3}}\omega_{\parallel\mathbf{4}})^{1/2}}$$
$$\times\left[\frac{M^2\gamma^2}{16\chi_\parallel^2} + \frac{1}{4}(\omega_{\parallel\mathbf{1}}\omega_{\parallel\mathbf{4}} - c_\parallel^2\mathbf{1}\cdot\mathbf{4}) + \frac{1}{4}(\omega_{\parallel\mathbf{2}}\omega_{\parallel\mathbf{3}} - c_\parallel^2\mathbf{2}\cdot\mathbf{3})\right], \quad (23)$$
$$D_{\mathrm{II}}(\mathbf{1234}) = \frac{\hbar^2}{V}\frac{\gamma^2}{8\chi_\parallel(\omega_{\parallel\mathbf{1}}\omega_{\parallel\mathbf{2}}\omega_{\parallel\mathbf{3}}\omega_{\parallel\mathbf{4}})^{1/2}}$$
$$\times\left[\frac{M^2\gamma^2}{16\chi_\parallel^2} - \frac{1}{4}(\omega_{\parallel\mathbf{1}}\omega_{\parallel\mathbf{4}} - c_\parallel^2\mathbf{1}\cdot\mathbf{4}) - \frac{1}{4}(\omega_{\parallel\mathbf{2}}\omega_{\parallel\mathbf{3}} - c_\parallel^2\mathbf{2}\cdot\mathbf{3})\right],$$
$$E(\mathbf{1234}) = \frac{\hbar^2}{V}\frac{\gamma^2}{8\chi_\parallel(\omega_{\parallel\mathbf{1}}\omega_{\parallel\mathbf{2}})^{1/2}}\left[\frac{\gamma M}{4\chi_\parallel} + \frac{\chi_\parallel}{2\gamma M}(\omega_{\parallel\mathbf{1}}\omega_{\parallel\mathbf{2}} - c_\parallel^2\mathbf{1}\cdot\mathbf{2}) + \frac{a_\perp}{2}(\mathbf{1}+\mathbf{4})\cdot(\mathbf{2}+\mathbf{3})\right].$$

All the elements of the expansion of the Hamiltonian contain products of even numbers of the $a_{\mathbf{k}}^{(\dagger)}$ operators.

Transforming the density of the magnetic moment (5) into

$$\boldsymbol{m} = \frac{1}{a''}\left[\frac{\pi_z(1+\varphi^2)}{M} - \frac{2a'MH}{a''}\left(1-2\frac{\varphi_x^2+\varphi_y^2}{1+\varphi^2}\right)\right]\boldsymbol{M} + \hat{\chi}\boldsymbol{H}, \quad (24)$$

we write its components for $\boldsymbol{H}=0$ in the secondary canonical variables as

$$m_x = \frac{\gamma}{2^{1/2}}\left(-\Theta Q_1Q_2 + \frac{\Theta}{2}\frac{Q_2\Pi_2Q_2}{\Pi_1} - \frac{1}{\Theta}\Pi_1\Pi_2\right), \quad (25a)$$
$$m_y = \frac{\gamma}{2^{1/2}}\left[-\frac{1}{\Theta 2}(Q_1\Pi_1+\Pi_1Q_1)\Pi_2 + \frac{1}{\Theta 2}\Pi_2Q_2\Pi_2 + \Theta Q_2\right], \quad (25b)$$
$$m_z = \frac{\gamma}{2}\left[\Pi_1 + Q_1\Pi_1Q_1 - \frac{\Theta^2}{2}\frac{Q_2^2}{\Pi_1} - \frac{1}{\Theta^2 2}\Pi_1\Pi_2^2 - \frac{1}{2}Q_1(Q_2\Pi_2+\Pi_2Q_2) + \frac{1}{4}\frac{Q_2\Pi_2^2Q_2}{\Pi_1}\right]. \quad (25c)$$

After the quantization, they satisfy the commutation relations generated by the algebra

$$[m_i(\mathbf{x},t), m_j(\mathbf{x}',t')] = \mathrm{i}\hbar\gamma\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')\varepsilon_{ijk}m_k(\mathbf{x},t). \quad (26)$$

In order to estimate small parameters for further calculations, let us evaluate the magnon contribution to the magnetization as a function of temperature:

$$\langle m_z\rangle_{\mathrm{o}} \simeq M\left[1 - \frac{\hbar}{V}\frac{1}{\Theta}\sum_{\mathbf{q}}\langle a_{\mathbf{q}}^\dagger a_{\mathbf{q}}\rangle + \frac{\hbar}{V}\frac{1}{2a'M^2}\sum_{\mathbf{q}}\frac{1}{\omega_{\parallel\mathbf{q}}}\langle b_{\mathbf{q}}^\dagger b_{\mathbf{q}}\rangle\right]$$
$$= M[1-(T/\theta_{\mathrm{c}})^{3/2} + (T/\theta_{\mathrm{o}})^2]. \quad (27)$$

Here $\langle\cdots\rangle_{\mathrm{o}}$ denotes an average with the density operator $\mathrm{e}^{-\mathcal{H}_{\mathrm{o}}/T}$. There are two characteristic temperatures in the expression $\theta_{\mathrm{c}}^{3/2} \simeq \hbar^{1/2}J_{0\perp}a_\perp^{1/2}/l_0$, $\theta_{\mathrm{o}}^2 \simeq \hbar c_\parallel J_{0\parallel}/l_0$, where $J_0 = J_{0\perp} + J_{0\parallel}$

is the energy of the exchange interaction between two spins separated from each other by the average distance $l_0$. We interpret $\theta_c$ and $\theta_o$ as the maximal shear and longitudinal magnon energy, which are of the order of the critical temperature.

We assume for the strongly magnetized system in which the magnetization decreases with temperature that $(T/\theta_c)^{3/2} > (T/\theta_o)^2$. From this inequality one may estimate the relation between the temperature and the parameter $\gamma M/\chi_\parallel$. The bilinear part of the fluctuation of the interaction energy for RCF may be written as $\varphi_{,i}\hat{\rho}\varphi_{,i}$, [5], where $\rho_{ij} = \rho_\perp\delta_{ij} + (\rho_\parallel - \rho_\perp)n_i n_j$ is called the spin stiffness [20]. This leads to $\theta_c = (\hbar\gamma/M)^{1/3}\rho_\perp$, $\theta_o = (\gamma\hbar)^{1/2}\chi_\parallel^{-1/4}\rho_\parallel^{3/4}$. For the strong-magnetization case, the assumption $\rho_\perp \gg \rho_\parallel$ leads to $\frac{\hbar\gamma M}{\chi_\parallel} \gg \frac{\hbar\gamma M}{\chi_\parallel}\frac{\rho_\parallel}{\rho_\perp} = \hbar c_\parallel^2/a_\perp > T$ via $(T/\theta_c)^{3/2} > (T/\theta_o)^2$. On the basis of these considerations we determine the small parameters for calculations as $T/\theta_c$, $\hbar\omega_{\perp\mathbf{k}}/\theta_c$, $\hbar\omega_{\parallel\mathbf{k}}/\theta_c$, $T/\theta_o$, $\hbar\omega_{\perp\mathbf{k}}/\theta_o$, $\hbar\omega_{\parallel\mathbf{k}}/\theta_o$, $T/\frac{\hbar\gamma M}{\chi_\parallel}$, $\omega_{\perp\mathbf{k}}/\frac{\gamma M}{\chi_\parallel}$ and $\omega_{\parallel\mathbf{k}}/\frac{\gamma M}{\chi_\parallel}$.

The linear response functions to an external magnetic field are related to the retarded Green functions of the components of the magnetic moment density:

$$\chi_{ij}(\mathbf{k},\omega) = -\frac{1}{\hbar}\langle\langle m_i(\mathbf{k},t), m_j^\dagger(\mathbf{k},0)\rangle\rangle_\omega^{(r)}. \tag{28}$$

Here $\langle\langle A(\mathbf{k},t), B(\mathbf{k},0)\rangle\rangle_\omega^{(r)}$ denotes the Fourier transform of the average $-\mathrm{i}\theta(t)\langle[A(\mathbf{k},t), B(\mathbf{k},0)]\rangle$. We neglect the index $(r)$ in studying the susceptibility in the vicinity of the spin-wave resonance. In order to evaluate the linear response functions, we use the perturbation method [21] outlined in appendix A.

### 3.2. Linear response functions

Let us study the components of the diagonalized tensor of the dynamical susceptibility $\hat{\chi}(\mathbf{k},\omega)$, the poles of which correspond to spin-wave frequencies [22]. In the vicinity of resonance, for $|\omega \pm \omega_{\mathbf{k}\perp}| \ll \omega_{\mathbf{k}\perp}$, $|\omega \pm \omega_{\mathbf{k}\parallel} \ll \omega_{\mathbf{k}\parallel}$, they can be divided into a dominant singular part and a non-singular part. We do not consider multi-magnon bound states, assuming that their energies are higher than the magnon energies in a wide wavelength region [23]. In calculations we will expand (25*a*)–(25*c*) up to the third order of expansion in fields and momenta.

The linear response of a FSG, for $\boldsymbol{H} = 0$, is described by the susceptibility tensor. Its components satisfy $\chi_{xy}(\mathbf{k},\omega) = \chi_{yx}^*(\mathbf{k},-\omega)$, which is a consequence of the general symmetry properties of Green functions for Hermitian operators, and $\chi_{xz} = \chi_{yz} = \chi_{zx} = \chi_{zy} = 0$ because all perturbation elements of these components are averages of the products of odd numbers of $a_\mathbf{k}^{(\dagger)}$ operators.

I intend to write the dynamical susceptibility using one-particle functions $G_{\alpha\beta}(\mathbf{k},\omega) = \langle\langle b_{\mathbf{k}\alpha}(t), b_{\mathbf{k}\beta}^\dagger\rangle\rangle_\omega$, $(\boldsymbol{b}_\mathbf{k} = (a_\mathbf{k}, b_\mathbf{k}, a_{-\mathbf{k}}^\dagger, b_{-\mathbf{k}}^\dagger))$. The structure of $\hat{G}(\mathbf{k},\omega)$ is determined by the structure of the mass operator

$$\hat{\Sigma}(\mathbf{k},\omega) = \begin{pmatrix} \Sigma_1 & 0 & \Sigma_3^+ & 0 \\ 0 & \Sigma_2 & 0 & \Sigma_4^+ \\ \Sigma_3 & 0 & \Sigma_1^+ & 0 \\ 0 & \Sigma_4 & 0 & \Sigma_2^+ \end{pmatrix}, \tag{29}$$

(here $\Sigma_i^+(\mathbf{k},\omega) \equiv \Sigma_i^*(-\mathbf{k},-\omega)$), according to the Dyson equation (A.3). The solution of the Dyson equation for the diagonal unperturbed one-particle functions is the matrix

$$\hat{G}(\mathbf{k},\omega) = \begin{pmatrix} G_1 & 0 & G_3 & 0 \\ 0 & G_2 & 0 & G_4 \\ G_3^+ & 0 & G_1^+ & 0 \\ 0 & G_4^+ & 0 & G_2^+ \end{pmatrix}, \tag{30}$$

where $G_i^+(\mathbf{k},\omega) \equiv G_i^*(-\mathbf{k},-\omega)$. Assuming a weak interaction between magnons, in the vicinity of the spin-wave resonance ($|\omega \pm \omega_{\perp\mathbf{k}}| \ll \omega_{\perp\mathbf{k}}, |\omega \pm \omega_{\|\mathbf{k}}| \ll \omega_{\|\mathbf{k}}$) we write the Green functions up to the first order in $\hat{\Sigma}(\mathbf{k},\omega)/\omega$ as

$$\begin{aligned} G_1 &\simeq G_{o1}, \\ G_2 &\simeq G_{o2} \\ G_3 &\simeq G_{o1}\Sigma_3^+ G_{o1}^+, \\ G_4 &\simeq G_{o2}\Sigma_4^+ G_{o2}^+, \end{aligned} \tag{31}$$

where

$$\begin{aligned} G_{o1} &= 1/(\omega - \omega_{\mathbf{k}\perp} - \Sigma_1), \qquad & G_{o2} &= 1/(\omega - \omega_{\mathbf{k}\|} - \Sigma_2), \\ G_{o1}^+ &= -1/(\omega + \omega_{\mathbf{k}\perp} + \Sigma_1^+), \qquad & G_{o2}^+ &= -1/(\omega + \omega_{\mathbf{k}\|} + \Sigma_2^+). \end{aligned} \tag{32}$$

The one-particle functions of the type $\langle\langle b_{\mathbf{k}\alpha}(t), A(0)\rangle\rangle$ in the expansion of the components of $\hat{\chi}(\mathbf{k},\omega)$ may be decomposed as a product of the matrix (30) and a four-component vector $\mathbf{\Lambda}_A(\mathbf{k},\omega)$, according to (A.7). Here $A$ denotes the element of the expansion of (25*a*) and (25*c*) up to the third order in fields and momenta. The structure of the vectors of the coefficients of decomposition (A.8) is as follows: six of them, indexed by the corresponding operators of the expansion of (25*a*) and (25*b*), $\mathbf{\Lambda}_{(Q_1Q_2)^\dagger}(\mathbf{k},\omega)$, $\mathbf{\Lambda}_{(\Pi_1'\Pi_2)^\dagger}(\mathbf{k},\omega)$, $\mathbf{\Lambda}_{(Q_1\Pi_2)^\dagger}(\mathbf{k},\omega)$, $\mathbf{\Lambda}_{(Q_2\Pi_2Q_2)^\dagger}(\mathbf{k},\omega)$, $\mathbf{\Lambda}_{(\Pi_2Q_2\Pi_2)^\dagger}(\mathbf{k},\omega)$, $\mathbf{\Lambda}_{[(Q_1\Pi_1'+\Pi_1'Q_1)\Pi_2]^\dagger}(\mathbf{k},\omega)$, take a similar form with the second and fourth component equal to zero:

$$\begin{aligned} \mathbf{\Lambda}_{(Q_1Q_2)^\dagger}(\mathbf{k},\omega) &= (\Lambda_{(Q_1Q_2)^\dagger}, 0, \Lambda^+_{(Q_1Q_2)^\dagger}, 0), \\ \mathbf{\Lambda}_{(\Pi_1'\Pi_2)^\dagger}(\mathbf{k},\omega) &= (\Lambda_{(\Pi_1'\Pi_2)^\dagger}, 0, \Lambda^+_{(\Pi_1'\Pi_2)^\dagger}, 0), \end{aligned} \tag{33}$$

etc, where $\Lambda_A^+(\mathbf{k},\omega) \equiv \Lambda_A^*(-\mathbf{k},-\omega)$. The other five, corresponding to operators of the expansion of (25*c*), $\mathbf{\Lambda}_{(Q_1^2)^\dagger}(\mathbf{k},\omega)$, $\mathbf{\Lambda}_{(Q_2^2+\frac{1}{\Theta^2}\Pi_2^2)^\dagger}(\mathbf{k},\omega)$, $\mathbf{\Lambda}_{(Q_1\Pi_1'Q_1)^\dagger}(\mathbf{k},\omega)$, $\mathbf{\Lambda}_{[\Pi_1'(Q_2^2-\frac{1}{\Theta^2}\Pi_2^2)]^\dagger}(\mathbf{k},\omega)$, $\mathbf{\Lambda}_{[Q_1(Q_2\Pi_2+\Pi_2Q_2)]^\dagger}(\mathbf{k},\omega)$, take the form

$$\mathbf{\Lambda}_{(Q_1^2)^\dagger}(\mathbf{k},\omega) = (0, \Lambda_{(Q_1^2)^\dagger}, 0, \Lambda^+_{(Q_1^2)^\dagger}), \tag{34}$$

etc (with the first and third components equal to zero).

The elements of the mass operator have been calculated up to the second order of the interaction. Relevant terms of the real and imaginary parts of them are presented in appendix B along with the relevant terms of the functions $\Lambda_A^{(+)}(\mathbf{k},\omega)$ calculated up to the first order in the interaction.

The singular part of $\hat{\chi}(\mathbf{k},\omega)$ contains all the one-particle functions and elements of the three-particle functions denoted by $W_{ij}(\mathbf{k},\omega)$. Decomposing the one-particle functions with (A.7) we find the singular part of the susceptibility in the form

$$\begin{aligned} \chi_{xx}(\mathbf{k},\omega) = -\frac{\gamma^2}{\hbar}\Bigg\{ &\Bigg[ -\frac{\hbar\Theta}{4} + \mathrm{i}\hbar^{1/2}2^{-1/2}\Theta^{3/2}\Lambda_{(Q_1Q_2)^\dagger}(\mathbf{k},\omega) + \mathrm{i}\hbar^{1/2}(2\Theta)^{-1/2}\Lambda_{(\Pi_1'\Pi_2)^\dagger}(\mathbf{k},\omega) \\ &- \mathrm{i}(\hbar\Theta)^{1/2}2^{-3/2}\Lambda_{(Q_2\Pi_2Q_2)^\dagger}(\mathbf{k},\omega)\Bigg][-G_1(\mathbf{k},\omega) + G_3^*(-\mathbf{k},-\omega)] \\ &+ \Bigg[\frac{\hbar\Theta}{4} + \mathrm{i}\hbar^{1/2}2^{-1/2}\Theta^{3/2}\Lambda^*_{(Q_1Q_2)^\dagger}(-\mathbf{k},-\omega) + \mathrm{i}\hbar^{1/2}(2\Theta)^{-1/2}\Lambda^*_{(\Pi_1'\Pi_2)^\dagger}(-\mathbf{k},-\omega) \\ &- \mathrm{i}(\hbar\Theta)^{1/2}2^{-3/2}\Lambda^*_{(Q_2\Pi_2Q_2)^\dagger}(-\mathbf{k},-\omega)\Bigg] \\ &\times [G_1^*(-\mathbf{k},-\omega) - G_3(\mathbf{k},\omega)] + W_{xx}(\mathbf{k},\omega)\Bigg\}, \end{aligned} \tag{35a}$$

$$\chi_{yy}(\mathbf{k},\omega) = -\frac{\gamma^2}{\hbar}\Bigg\{\Bigg[-\frac{\hbar\Theta}{4} + \hbar^{1/2}2^{-1/2}\Theta^{1/2}\Lambda_{(Q_1\Pi_2)^\dagger}(\mathbf{k},\omega)$$
$$+ \hbar^{1/2}\Theta^{-1/2}2^{-3/2}\Lambda_{[(Q_1\Pi_1'+\Pi_1'Q_1)\Pi_2]^\dagger}(\mathbf{k},\omega)$$
$$- \hbar^{1/2}\Theta^{-1/2}2^{-3/2}\Lambda_{(\Pi_2Q_2\Pi_2)^\dagger}(\mathbf{k},\omega)\Bigg][-G_1(\mathbf{k},\omega) - G_3^*(-\mathbf{k},-\omega)]$$
$$+ \Bigg[-\frac{\hbar\Theta}{4} + \hbar^{1/2}2^{-1/2}\Theta^{1/2}\Lambda^*_{(Q_1\Pi_2)^\dagger}(-\mathbf{k},-\omega)$$
$$+ \hbar^{1/2}\Theta^{-1/2}2^{-3/2}\Lambda^*_{[(Q_1\Pi_1'+\Pi_1'Q_1)\Pi_2]^\dagger}(-\mathbf{k},-\omega)$$
$$- \hbar^{1/2}\Theta^{-1/2}2^{-3/2}\Lambda^*_{(\Pi_2Q_2\Pi_2)^\dagger}(-\mathbf{k},-\omega)\Bigg]$$
$$\times [-G_1^*(-\mathbf{k},-\omega) - G_3(\mathbf{k},\omega)] + W_{yy}(\mathbf{k},\omega)\Bigg\}, \tag{35b}$$

$$\chi_{xy}(\mathbf{k},\omega) = \chi_{yx}^*(\mathbf{k},-\omega) = -\frac{\gamma^2}{\hbar}\Bigg\{\Bigg[-\frac{\mathrm{i}\hbar\Theta}{4} + \mathrm{i}(\hbar\Theta)^{1/2}2^{-3/2}\Lambda_{(Q_1\Pi_2)^\dagger}(\mathbf{k},\omega)$$
$$- \mathrm{i}\hbar^{1/2}2^{-5/2}\Theta^{-1/2}\Lambda_{(\Pi_2Q_2\Pi_2)^\dagger}(\mathbf{k},\omega)\Bigg][-G_1(\mathbf{k},\omega) + G_3^*(-\mathbf{k},-\omega)]$$
$$+ \Bigg[-\frac{\mathrm{i}\hbar\Theta}{4} + \mathrm{i}(\hbar\Theta)^{1/2}2^{-3/2}\Lambda^*_{(Q_1\Pi_2)^\dagger}(-\mathbf{k},-\omega)$$
$$- \mathrm{i}\hbar^{1/2}2^{-5/2}\Theta^{-1/2}\Lambda^*_{(\Pi_2Q_2\Pi_2)^\dagger}(-\mathbf{k},-\omega)\Bigg][G_1^*(-\mathbf{k},-\omega) - G_3(\mathbf{k},\omega)]$$
$$+ [-\hbar^{1/2}2^{-3/2}\Theta^{3/2}\Lambda_{(Q_1Q_2)^\dagger}(\mathbf{k},\omega) - \hbar^{1/2}2^{-3/2}\Theta^{-1/2}\Lambda_{(\Pi_1'\Pi_2)^\dagger}(\mathbf{k},\omega)$$
$$+ (\hbar\Theta)^{1/2}2^{-5/2}\Lambda_{(Q_2\Pi_2Q_2)^\dagger}(\mathbf{k},\omega)][G_1(k,\omega) + G_3^*(-\mathbf{k},-\omega)]$$
$$+ [-\hbar^{1/2}2^{-3/2}\Theta^{3/2}\Lambda^*_{(Q_1Q_2)^\dagger}(-\mathbf{k},-\omega) - \hbar^{1/2}2^{-3/2}\Theta^{-1/2}\Lambda^*_{(\Pi_1'\Pi_2)^\dagger}(-\mathbf{k},-\omega)$$
$$+ (\hbar\Theta)^{1/2}2^{-5/2}\Lambda^*_{(Q_2\Pi_2Q_2)^\dagger}(-\mathbf{k},-\omega)][G_1^*(-\mathbf{k},-\omega)$$
$$+ G_3(\mathbf{k},\omega)] + W_{xy}(\mathbf{k},\omega)\Bigg\}, \tag{35c}$$

$$\chi_{zz}(\mathbf{k},\omega) = -\frac{\gamma^2}{4\hbar}\{[-\hbar a'M^2\omega_{\|\mathbf{k}} + \mathrm{i}\hbar^{1/2}2\Theta(a'M^2\omega_{\|\mathbf{k}})^{1/2}\Lambda_{(Q_1^2)^\dagger}(\mathbf{k},\omega)$$
$$+ \mathrm{i}\hbar^{1/2}2(a'M^2\omega_{\|\mathbf{k}})^{1/2}\Lambda_{(Q_1\Pi_1'Q_1)^\dagger}(\mathbf{k},\omega)][-G_2(\mathbf{k},\omega) + G_4^*(-\mathbf{k},-\omega)]$$
$$+ [\hbar aM^2\omega_{\|\mathbf{k}} + \mathrm{i}\hbar^{1/2}2\Theta(a'M^2\omega_{\|\mathbf{k}})^{1/2}\Lambda^*_{(Q_1^2)^\dagger}(-\mathbf{k},-\omega) + \mathrm{i}\hbar^{1/2}2(a'M^2\omega_{\|\mathbf{k}})^{1/2}$$
$$\times \Lambda^*_{(Q_1\Pi_1'Q_1)^\dagger}(-\mathbf{k},-\omega)][G_2^*(-\mathbf{k},-\omega) - G_4(\mathbf{k},\omega)] + W_{zz}(\mathbf{k},\omega)\}, \tag{35d}$$

where the functions $W_{ij}(\mathbf{k},\omega)$ containing singular parts of the many-particle functions are of higher order in $T/\theta_c, \hbar\omega_{\perp\mathbf{k}}/\theta_c, \ldots$, than the contributions from the one-particle functions.

Let us define $\Delta_\perp = \omega_{\mathbf{k}\perp}|_{\mathbf{k}=0} = \gamma(\alpha_1+\alpha_2)/(2M)$, $\Delta_\| = \omega_{\mathbf{k}\|}|_{\mathbf{k}=0} = [\gamma^2\alpha_1/(2\chi_\|)]^{1/2}$. Calculating the functions $\Lambda_A(\mathbf{k},\omega)$ in (35*a*)–(35*d*) for $\hbar\gamma H, \hbar\Delta_{\perp,\|} \ll \hbar\omega_{\perp,\|\mathbf{k}}, T$ one finds

$$\chi_{xx}(\mathbf{k},\omega) = \chi_{yy}(\mathbf{k},\omega) = -\frac{\gamma M}{2}[1-(T/\theta_c)^{3/2}][(\omega-\omega_{\perp\mathbf{k}}-\delta\omega_{\perp\mathbf{k}}+\mathrm{i}\Gamma_{\perp\mathbf{k}})^{-1}$$
$$- (\omega+\omega_{\perp\mathbf{k}}+\delta\omega_{\perp\mathbf{k}}+\mathrm{i}\Gamma_{\perp\mathbf{k}})^{-1}], \tag{36a}$$

$$\chi_{xy}(\mathbf{k},\omega) = \chi_{yx}^*(\mathbf{k},-\omega) = -\mathrm{i}\frac{\gamma M}{2}[1+(T/\theta_c)^{3/2}][(\omega-\omega_{\perp\mathbf{k}}-\delta\omega_{\perp\mathbf{k}}+\mathrm{i}\Gamma_{\perp\mathbf{k}})^{-1}$$
$$+ (\omega+\omega_{\perp\mathbf{k}}+\delta\omega_{\perp\mathbf{k}}+\mathrm{i}\Gamma_{\perp\mathbf{k}})^{-1}], \tag{36b}$$

$$\chi_{zz}(\mathbf{k},\omega) = -\frac{\chi_\parallel}{2}[1+(T/\theta_o)^2]\omega_{\parallel\mathbf{k}}[(\omega-\omega_{\parallel\mathbf{k}}-\delta\omega_{\parallel\mathbf{k}} + \mathrm{i}\Gamma_{\parallel\mathbf{k}})^{-1} - (\omega+\omega_{\parallel\mathbf{k}}+\delta\omega_{\parallel\mathbf{k}}+\mathrm{i}\Gamma_{\parallel\mathbf{k}})^{-1}]. \quad (36c)$$

Here $\delta\omega_{\perp\mathbf{k}} = \mathrm{Re}\,\Sigma_1(\mathbf{k},\omega_{\perp\mathbf{k}})$, $\Gamma_{\perp\mathbf{k}} = -\mathrm{Im}\,\Sigma_1(\mathbf{k},\omega_{\perp\mathbf{k}})$, $\delta\omega_{\parallel\mathbf{k}} = \mathrm{Re}\,\Sigma_2(\mathbf{k},\omega_{\parallel\mathbf{k}})$, $\Gamma_{\parallel\mathbf{k}} = -\mathrm{Im}\,\Sigma_2(\mathbf{k},\omega_{\parallel\mathbf{k}})$. In the case of a strong external magnetic field $\hbar\gamma H \gg T$, the temperature corrections to $\chi_{xx}$, $\chi_{xy}$ are negligible because they are proportional to $\mathrm{e}^{-\hbar\gamma H/T}$.

The poles of the diagonalized susceptibility tensor components:

$$\begin{aligned} \chi_\perp^+ &= \chi_{xx} + \mathrm{i}\chi_{xy}, \\ \chi_\perp^- &= \chi_{xx} - \mathrm{i}\chi_{xy}, \\ \chi_\parallel &= \chi_{zz} \end{aligned} \quad (37)$$

are determined by the poles of the one-particle functions, so one can find the perturbation corrections to the spin-wave frequencies and their damping coefficients studying diagonal elements of the mass operator $\Sigma_1(\mathbf{k},\omega)$ and $\Sigma_2(\mathbf{k},\omega)$. We evaluate contributions to them coming from the triple-process part of the interaction $\mathcal{H}_3$ (denoted $\delta\omega^3_{\perp\mathbf{k}}$, $\delta\omega^3_{\parallel\mathbf{k}}$, $\Gamma^3_{\perp\mathbf{k}}$, $\Gamma^3_{\parallel\mathbf{k}}$) and the contributions from the $\mathcal{H}_4$ interaction part (denoted $\delta\omega^4_{\perp\mathbf{k}}$, $\delta\omega^4_{\parallel\mathbf{k}}$, $\Gamma^4_{\perp\mathbf{k}}$, $\Gamma^4_{\parallel\mathbf{k}}$). Additional indices (1) and (2) at the top of these symbols denote contributions of the first and second order in the interaction. From (B.1)–(B.4) one finds

$$\delta\omega_{\perp\mathbf{k}} = \delta\omega^{3(2)}_{\perp\mathbf{k}} + \delta\omega^{4(1)}_{\perp\mathbf{k}} \approx 0.5\frac{\gamma M}{\chi_\parallel}[(T/\theta_c)^{3/2} + (T/\theta_o)^2] \quad (38a)$$

$$\delta\omega_{\parallel\mathbf{k}} = \delta\omega^{3(2)}_{\parallel\mathbf{k}} + \delta\omega^{4(1)}_{\parallel\mathbf{k}} \approx 0.25\frac{\gamma M}{\chi_\parallel}\left(\frac{\gamma M}{\chi_\parallel}\Big/\omega_{\parallel k}\right)[(T/\theta_c)^{3/2} + (T/\theta_o)^2] \quad (38b)$$

$$\Gamma^{3(2)}_{\perp\mathbf{k}} \sim \frac{c_\parallel^2}{a_\perp}\left(\frac{c_\parallel^2}{a_\perp}\Big/\frac{\gamma M}{\chi_\parallel}\right)(\hbar a_\perp \mathrm{k}^2/\theta_c)^{1/2}(T/\theta_c)\mathrm{e}^{-\frac{\hbar c_\parallel^2}{a_\perp}/T}\sinh(2\hbar c_\parallel k/T) \quad (39a)$$

$$\Gamma^{3(2)}_{\parallel\mathbf{k}} \sim 10^{-1}\frac{c_\parallel^2}{a_\perp}\left(\frac{c_\parallel^2}{a_\perp}\Big/\frac{\gamma M}{\chi_\parallel}\right)\left(\frac{\hbar c_\parallel^2}{a_\perp}\Big/\theta_c\right)^{1/2}(T/\theta_c)\mathrm{e}^{-\frac{\hbar c_\parallel^2}{4a_\perp}/T}\sinh(2\hbar c_\parallel k/T) \quad (39b)$$

for the long-wavelength regime $\hbar\Delta_{\perp,\parallel} \ll T, \hbar\omega_{\mathbf{k}\perp,\parallel}$ and

$$\Gamma^{4(2)}_{\perp\mathbf{k}} \sim \begin{cases} 10^{-1}\dfrac{\gamma M}{\chi_\parallel}\left(\dfrac{\hbar\gamma M}{\chi_\parallel}\Big/\theta_c\right)(T/\theta_c)^2 & \hbar\Delta_{\perp,\parallel} \ll \hbar\omega_{\mathbf{k}\perp,\parallel} \ll T \\ 10^{-1}\dfrac{\gamma M}{\chi_\parallel}(\hbar a_\perp \mathrm{k}^2/\theta_c)^{1/2}\left(\dfrac{\hbar\gamma M}{\chi_\parallel}\Big/\theta_c\right)(T/\theta_c)^{3/2} & \hbar\Delta_{\perp,\parallel} \ll T \ll \hbar\omega_{\mathbf{k}\perp,\parallel} \\ 10^{-1}\dfrac{\gamma M}{\chi_\parallel}(\hbar a_\perp \mathrm{k}^2/\theta_c)^{1/2}\left(\dfrac{\hbar\gamma M}{\chi_\parallel}\Big/\theta_c\right) & \\ \quad\times(T/\theta_c)^{3/2}\mathrm{e}^{-\hbar\Delta_\perp/T} & T \ll \hbar\Delta_{\perp,\parallel} \ll \hbar\omega_{\mathbf{k}\perp,\parallel} \end{cases} \quad (40a)$$

$$\Gamma^{4(2)}_{\parallel\mathbf{k}} \sim \begin{cases} 10^{-1}T/\hbar(T/\hbar\omega_{\mathbf{k}\parallel})\left(\dfrac{\gamma M}{\chi_\parallel}\Big/\dfrac{c_\parallel^2}{a_\perp}\right)^3\left(\dfrac{\gamma M}{\chi_\parallel}\Big/\theta_c\right)^3 & \hbar\Delta_{\perp,\parallel} \ll \hbar\omega_{\mathbf{k}\perp,\parallel}, T \\ 10^{-1}(T/\hbar)\left(\dfrac{\gamma M}{\chi_\parallel}\Big/\dfrac{c_\parallel^2}{a_\perp}\right)^3\left(\dfrac{\hbar\gamma M}{\chi_\parallel}\Big/\theta_c\right)^3 & \\ \quad\times(T/\hbar\omega_{\mathbf{k}\parallel})(\hbar\Delta_\parallel/T)^{1/2}\mathrm{e}^{-\hbar\Delta_\parallel/T} & T \ll \hbar\Delta_{\perp,\parallel} \ll \hbar\omega_{\mathbf{k}\perp,\parallel}. \end{cases} \quad (40b)$$

Note that for $\Delta_\parallel = \Delta_\perp = 0$ the frequency corrections $\delta\omega_{\perp\mathbf{k}}$, $\delta\omega_{\parallel\mathbf{k}}$ are different from zero in the long-wavelength limit $\mathbf{k} \to 0$, which possibility has been predicted in the earlier section. The main contributions to the damping coefficients come from $\Gamma^{4(2)}_{\perp\mathbf{k}}$, $\Gamma^{4(2)}_{\parallel\mathbf{k}}$ since $\Gamma^{3(2)}_{\perp\mathbf{k}}$, $\Gamma^{3(2)}_{\parallel\mathbf{k}}$ are proportional to small exponential factors.

The method of evaluating the sum (B.4) (the expression for the dominant contribution to Im $\Sigma_1(\mathbf{k},\omega)$) was developed in [24, 25] where details of the calculations are available. It is important to mention, however, that the summary scattering amplitude (B.5) in Im $\Sigma_1^{4(2)}(\mathbf{k},\omega)$ (B.4) takes the form

$$M_\perp(\mathbf{p},\mathbf{s},\mathbf{k}+\mathbf{p}-\mathbf{s},\mathbf{k}) \propto \frac{\gamma M}{\chi_\parallel} - 2a_\perp|\mathbf{p}+\mathbf{k}|^2 \tag{41}$$

on a mass surface defined by the equation $\omega_{\perp\mathbf{k}} + \omega_{\perp\mathbf{p}} - \omega_{\perp\mathbf{s}} - \omega_{\perp\mathbf{k+p-s}} = 0$, which leads to the dominant contribution to the damping coefficient being of the form

$$\begin{aligned}\Gamma^{4(2)}_{\perp\mathbf{k}} = {} & \frac{\gamma M}{\chi_\parallel}\left(\frac{\hbar\gamma M}{\chi_\parallel}\Big/\theta_c\right)(T/\theta_c)^2 C + \omega_{\perp\mathbf{k}}\left(\frac{\hbar\gamma M}{\chi_\parallel}\Big/\theta_c\right)(T/\theta_c)^2 \\ & \times [C_{(1)} + C'_{(1)}\ln(T/\hbar a_\perp k^2) + C''_{(1)}\ln^2(T/\hbar a_\perp k^2)] + \omega_{\perp\mathbf{k}}(\hbar a_\perp k^2/\theta_c)(T/\theta_c)^2 \\ & \times [C_{(2)} + C'_{(2)}\ln(T/\hbar a_\perp k^2) + C''_{(2)}\ln^2(T/\hbar a_\perp k^2)],\end{aligned} \tag{42}$$

($C,\ldots,C''_{(2)}$ are constants) for the hydrodynamical regime $\hbar\Delta_\perp \ll \hbar\omega_{\perp\mathbf{k}} \ll T$. This dependence is different from the appropriate dependence for the Heisenberg ferromagnet [24, 26, 27] and from the result of the macroscopic approach to spin-wave interactions in ferromagnets [28] for which the expression for $\Gamma^{4(2)}_{\mathbf{k}}$ is similar to (B.4) but it contains a scattering amplitude which may be written as $M_\perp(\mathbf{p},\mathbf{s},\mathbf{k}+\mathbf{p}-\mathbf{s},\mathbf{k}) \propto \mathbf{p}\cdot\mathbf{k}$ on the mass surface. This leads to $\Gamma^{4(2)}_{\mathbf{k}} \propto k^4T^2[C + C'\ln(T/\hbar\omega_{\mathbf{k}}) + C''\ln^2(T/\hbar\omega_{\mathbf{k}})]$ for the ferromagnet. Our result differs also from the result found in the framework of microscopic approaches to the diluted Heisenberg ferromagnet ($\Gamma_{\mathbf{k}} \propto k^5$, [29, 30]) not including the presence of magnetic anisotropy and non-collinearity, included in our model. The general hydrodynamic prediction applicable to perfect and diluted collinear ferromagnets gives $\Gamma_{\mathbf{k}} \propto k^4$, [20].

The wavevector dependence of $\Gamma_{\perp\mathbf{k}}$ different from the dependence for the Heisenberg ferromagnet was observed via the neutron scattering from FSGs $(Fe_{1-x}Mn_x)_{75}P_{16}B_6Al_3$ ($x \approx 0.25$) and $Fe_xCr_{1-x}$ ($x \approx 0.25$) [31, 32]. The possibility of the observation of the gap in the $\Gamma_{\perp\mathbf{k}}$ spectrum for $\mathbf{k} \to 0$ was suggested from fitting the experimental data of [32].

Note that, tending to the limit $\omega = 0$, $k \to 0$, one finds the magnon contribution to the static susceptibility:

$$\begin{aligned}\chi_{\perp st} &= M[1-(T/\theta_c)^{3/2}]\frac{1}{H + (\alpha_1+\alpha_2)/(2M)}, \\ \chi_{\parallel st} &= \chi_\parallel[1 + (T/\theta_o)^2].\end{aligned} \tag{43}$$

### *3.3. Correlation functions of magnetization and neutron scattering cross section*

The autocorrelation functions of the density of the magnetic moment components:

$$C_{m_im_i}(\mathbf{k},\omega) = \langle\delta m_i(\mathbf{k},t)\delta m_j(-\mathbf{k},0)\rangle_{-\omega} \tag{44}$$

($\delta\boldsymbol{m} = \boldsymbol{m} - \boldsymbol{M}_o$), which are the spectral-weight functions of the linear response, play an important role in various experimental methods, especially in the commonly used method of spin-wave analysing the inelastic neutron scattering that enables one to verify our results. Let us evaluate the correlation functions of the magnetization components as well as the inelastic neutron scattering cross section depending on them. We calculate them to the lowest order in $T/\theta_c$, $\hbar\omega_{\mathbf{k}\perp}/\theta_c$, ..., analogously to the earlier considerations for GSG and for the Heisenberg antiferromagnet [12, 25]. From the fluctuation-dissipation theorem, the autocorrelation functions are determined by the magnetic susceptibility via

$$C_{m_im_i}(\mathbf{k},\omega) = \hbar\,\mathrm{Im}\,\chi_{ii}(\mathbf{k},\omega)\coth(\hbar\omega/2T). \tag{45}$$

Let us denote $\Sigma''_\nu(\mathbf{k},\omega) \equiv \mathrm{Im}\,\Sigma_\nu(\mathbf{k},\omega)$, $\Lambda'_A(\mathbf{k},\omega) \equiv \mathrm{Re}\,\Lambda_A(\mathbf{k},\omega)$, $\Lambda''_A(\mathbf{k},\omega) \equiv \mathrm{Im}\,\Lambda_A(\mathbf{k},\omega)$, $W''_{ij}(\mathbf{k},\omega) \equiv \mathrm{Im}\,W_{ij}(\mathbf{k},\omega)$. Using (31), one obtains for $\hbar\Delta_\perp, \hbar\Delta_\parallel \ll \hbar\omega_{\perp\mathbf{k}}, \hbar\omega_{\parallel\mathbf{k}} \ll T$

$$\begin{aligned}\mathrm{Im}\,\chi_{xx}(\mathbf{k},\omega) = \mathrm{Im}\,\chi_{yy}(\mathbf{k},\omega) = \frac{\gamma M}{2}&(G_{o1}^2(\mathbf{k},\omega)\Sigma''_1(\mathbf{k},\omega) - G_{o1}^{+2}(\mathbf{k},\omega)\Sigma''_1(-\mathbf{k},-\omega)\\ &- G_{o1}(\mathbf{k},\omega)G_{o1}^{+}(\mathbf{k},\omega)\{-\Sigma''_3(\mathbf{k},\omega) + \Sigma''_3(-\mathbf{k},-\omega)\\ &+ (\omega+\omega_{\perp\mathbf{k}})[\hbar^{-1/2}\Theta^{1/2}2^{3/2}\Lambda'_{(Q_1Q_2)^\dagger}(\mathbf{k},\omega)\\ &+ \hbar^{-1/2}\Theta^{-3/2}2^{3/2}\Lambda'_{(\Pi'_1\Pi_2)^\dagger}(\mathbf{k},\omega) - (\hbar\Theta)^{-1/2}2^{1/2}\Lambda'_{(Q_2\Pi_2Q_2)^\dagger}(\mathbf{k},\omega)]\\ &+ (\omega-\omega_{\perp\mathbf{k}})[\hbar^{-1/2}2^{3/2}\Theta^{1/2}\Lambda'_{(Q_1Q_2)^\dagger}(-\mathbf{k},-\omega)\\ &+ \hbar^{-1/2}\Theta^{-3/2}2^{3/2}\Lambda'_{(\Pi'_1\Pi_2)^\dagger}(-\mathbf{k},-\omega)\\ &- (\hbar\Theta)^{-1/2}2^{1/2}\Lambda'_{(Q_2\Pi_2Q_2)^\dagger}(-\mathbf{k},-\omega)]\}) - \frac{\gamma^2}{\hbar}W''_{xx}(\mathbf{k},\omega),\end{aligned} \tag{46a}$$

$$\begin{aligned}\mathrm{Im}\,\chi_{zz}(\mathbf{k},\omega) = \frac{\chi}{2}\omega_{\parallel\mathbf{k}}&(G_{o2}^2(\mathbf{k},\omega)\Sigma''_2(\mathbf{k},\omega) - G_{o2}^{+2}(\mathbf{k},\omega)\Sigma''_2(-\mathbf{k},-\omega)\\ &- G_{o2}(\mathbf{k},\omega)G_{o2}^{+}(\mathbf{k},\omega)\{-\Sigma''_4(\mathbf{k},\omega) + \Sigma''_4(-\mathbf{k},-\omega)\\ &+ (\omega+\omega_{\parallel\mathbf{k}})[\hbar^{-1/2}2\Theta(a'M^2\omega_{\parallel\mathbf{k}})^{-1/2}\Lambda'_{(Q_1^2)^\dagger}(\mathbf{k},\omega)\\ &+ \hbar^{-1/2}2(a'M^2\omega_{\parallel\mathbf{k}})^{-1/2}\Lambda'_{(Q_1\Pi'_1Q_1)^\dagger}(\mathbf{k},\omega)]\\ &+ (\omega-\omega_{\parallel\mathbf{k}})[\hbar^{-1/2}2\Theta(a'M^2\omega_{\parallel\mathbf{k}})^{-1/2}\Lambda'_{(Q_1^2)^\dagger}(-\mathbf{k},-\omega)\\ &+ \hbar^{-1/2}2(a'M^2\omega_{\parallel\mathbf{k}})^{-1/2}\Lambda'_{(Q_1\Pi'_1Q_1)^\dagger}(-\mathbf{k},-\omega)]\}) - \frac{\gamma^2}{4\hbar}W''_{zz}(\mathbf{k},\omega).\end{aligned} \tag{46b}$$

We neglect the terms depending on the off-diagonal mass-operator elements $\Sigma_3(\mathbf{k},\omega)$ and $\Sigma_4(\mathbf{k},\omega)$ as well as the terms depending on the functions $\Lambda_A(\mathbf{k},\omega)$ since they are multiplied by the factors $(\omega^2-\omega_{\perp\mathbf{k}}^2)/\omega_{\perp\mathbf{k}}^2$ or $(\omega^2-\omega_{\parallel\mathbf{k}}^2)/\omega_{\parallel\mathbf{k}}^2$, which are small in the vicinity of resonances. We also neglect the $\mathrm{Im}\,\hat{\Sigma}^{3(2)}(\mathbf{k},\omega)$ contributions to the mass operator related to the $\mathcal{H}_3$ part of the interaction, because they are proportional to $\exp(-\hbar c_\parallel^2/a_\perp T)$. For $\hbar\Delta_\perp, \hbar\Delta_\parallel \ll \hbar\omega_{\perp\mathbf{k}}, \hbar\omega_{\parallel\mathbf{k}} \ll T$, near the resonance, we find

$$\begin{aligned}\Sigma_1^{4(2)''}(\mathbf{k},\omega) &= -\Gamma_{\perp\mathbf{k}}^{4(2)}\omega/\omega_{\perp\mathbf{k}},\\ \Sigma_2^{4(2)''}(\mathbf{k},\omega) &= -\Gamma_{\parallel\mathbf{k}}^{4(2)}\omega/\omega_{\parallel\mathbf{k}}.\end{aligned} \tag{47}$$

According to the method described in [25], a simple form of $\mathrm{Im}\,\hat{\Sigma}^{4(2)}(\mathbf{k},\omega)$ is caused by the fact that the main term of the amplitude $M_\perp(\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{k})$ in the expression in appendix B for $\mathrm{Im}\,\Sigma_1^{4(2)}(\mathbf{k},\omega)$ (B.4) is constant and the adequate term in the expression for $\mathrm{Im}\,\Sigma_2^{4(2)}(\mathbf{k},\omega)$ is equal to constant $\times\ (\omega_{\parallel\mathbf{1}}\omega_{\parallel\mathbf{2}}\omega_{\parallel\mathbf{3}}\omega_{\parallel\mathbf{k}})^{-1/2}$.

From (45), (46*a*) and (46*b*) for the long-wavelength region ($\hbar\omega \ll T$) one writes

$$\begin{aligned}C_{m_xm_x}(\mathbf{k},\omega) &= C_{m_ym_y}(\mathbf{k},\omega)\\ &= 2\gamma MT\frac{\Gamma_{\perp\mathbf{k}}^{4(2)}\omega_{\perp\mathbf{k}}[1+(\omega/\omega_{\perp\mathbf{k}})^2]}{[(\omega-\omega_{\perp\mathbf{k}}-\delta\omega_{\perp\mathbf{k}})^2+\Gamma_{\perp\mathbf{k}}^2][(\omega+\omega_{\perp\mathbf{k}}+\delta\omega_{\perp\mathbf{k}})^2+\Gamma_{\perp\mathbf{k}}^2]}\end{aligned} \tag{48a}$$

$$C_{m_zm_z}(\mathbf{k},\omega) = 2\chi_\parallel T\frac{\Gamma_{\parallel\mathbf{k}}^{4(2)}\omega_{\parallel\mathbf{k}}^2[1+(\omega/\omega_{\parallel\mathbf{k}})^2]}{[(\omega-\omega_{\parallel\mathbf{k}}-\delta\omega_{\parallel\mathbf{k}})^2+\Gamma_{\parallel\mathbf{k}}^2][(\omega+\omega_{\parallel\mathbf{k}}+\delta\omega_{\parallel\mathbf{k}})^2+\Gamma_{\parallel\mathbf{k}}^2]}. \tag{48b}$$

Let us evaluate the inelastic neutron scattering differential cross section from a FSG starting from the general expression for magnets [33, 34] which, up to a multiplicative constant, is of the form

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_{p'}\,\mathrm{d}\Omega} \propto \frac{p'}{p} F^2(\mathbf{k}) \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}/\hbar(E_{p'}-E_p)t} \sum_{\mu\nu} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}_\nu-\mathbf{x}_\mu)} \langle s_\mu^i(0) s_\nu^j(t)\rangle (\delta_{ij} - k_i k_j/k^2). \quad (49)$$

Here the magnetic form factor $F(\mathbf{k})$ is determined by the atomic structure and $\mathbf{k} = \mathbf{p} - \mathbf{p}'$, where $\mathbf{p}$, $\mathbf{p}'$ denote the initial and final momenta of the scattered neutron and $\boldsymbol{s}_\mu$ denotes the spin of the ion indicated by $\mu$. The spin vectors can be decomposed as

$$\boldsymbol{s}_\mu = s_{z\mu}\boldsymbol{n} + 2^{-1} s_{-\mu} \mathrm{e}^{-\mathrm{i}\mathbf{k}_0\cdot\mathbf{x}_\mu} \boldsymbol{n}_+ + 2^{-1} s_{+\mu} \mathrm{e}^{\mathrm{i}\mathbf{k}_0\cdot\mathbf{x}_\mu} \boldsymbol{n}_-, \quad (50)$$

($\boldsymbol{n} = \boldsymbol{M}_\mathrm{o}/M = (0,0,1)$, $\boldsymbol{n}_+ = (1,\mathrm{i},0)$, $\boldsymbol{n}_- = (1,-\mathrm{i},0)$, $s_+ = s_x + \mathrm{i}s_y$, $s_- = s_x - \mathrm{i}s_y$). For the structure vector $\mathbf{k}_0 \to 0$, using the identities $\gamma^2/V^2 \sum_{\mu\nu} \langle s_{\mp\mu} s_{\pm\nu}(t)\rangle \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}_\nu-\mathbf{x}_\mu)} = \langle (m_{x\mathbf{k}} \mp \mathrm{i}m_{y\mathbf{k}})(m_{x-\mathbf{k}}(t) \pm \mathrm{i}m_{y-\mathbf{k}}(t))\rangle = \langle m_\mp(\mathbf{k},0) m_\pm(\mathbf{k},t)\rangle$, $\gamma^2/V^2 \sum_{\mu\nu} \langle s_{z\mu} s_{z\nu}(t)\rangle \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}_\nu-\mathbf{x}_\mu)} = \langle m_{z\mathbf{k}}, m_{z-\mathbf{k}}(t)\rangle$, one finds the scattering intensity for the frequency region near the spin-wave resonance, in which the scattering via spin waves is dominant, in the form

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}\Omega\,\mathrm{d}E_{\mathbf{p}'}} \propto \frac{p'}{p} F'^2(\mathbf{k}) \{C_{m_z m_z}(\mathbf{k},\omega)[1 - (\boldsymbol{n}\mathbf{k}/k)^2] + C_{m_x m_x}(\mathbf{k},\omega)[1 + (\boldsymbol{n}\mathbf{k}/\mathrm{k})^2]\} \quad (51)$$

similar to the intensity for GSG in the presence of an external magnetic field [35].

## 4. Dynamics of a skewed spin glass

### *4.1. Quantization of Hamiltonian and magnetization*

The canonical momenta may be written as the components of the vector

$$\boldsymbol{\pi} = 2a \frac{\dot{\boldsymbol{\varphi}}(1+\varphi^2) - \boldsymbol{\varphi}(\boldsymbol{\varphi}\cdot\dot{\boldsymbol{\varphi}})}{(1+\varphi^2)^2} + (2a'\boldsymbol{M}\cdot\dot{\boldsymbol{\Omega}} + a'' + 2a'\boldsymbol{M}\cdot\boldsymbol{H}/a'')\boldsymbol{\mu} + 2a/a''\boldsymbol{\nu}, \quad (52)$$

where $\boldsymbol{\mu} \equiv \frac{M_\mathrm{o} + (\varphi\times M_\mathrm{o})}{1+\varphi^2}$ and $\boldsymbol{\nu} = \frac{\boldsymbol{H}+(\boldsymbol{H}\times\boldsymbol{\varphi})}{1+\varphi^2}$. Via

$$\boldsymbol{\eta} = (a'' + 2a'\boldsymbol{M}\cdot\boldsymbol{H}/a'')\boldsymbol{\mu} + 2a/a''\boldsymbol{\nu} \quad (53)$$

the Hamiltonian takes the form

$$\begin{aligned}\mathcal{H} = \frac{1}{4a}(\{|\boldsymbol{\pi}-\boldsymbol{\eta}|^2 + [(\boldsymbol{\pi}-\boldsymbol{\eta})\cdot\boldsymbol{\varphi}]^2\}(1+\varphi^2) \\ - \frac{a'}{a+a'M^2}\{(\boldsymbol{\pi}-\boldsymbol{\eta})\cdot[\boldsymbol{M}_\mathrm{o} + \boldsymbol{\varphi}\times\boldsymbol{M}_\mathrm{o} + (\boldsymbol{\varphi}\cdot\boldsymbol{M}_\mathrm{o})\boldsymbol{\varphi}]\}^2) \\ - b(\boldsymbol{M}\cdot\boldsymbol{\Omega}_{,i})^2 - b'\boldsymbol{\Omega}_{,i}\cdot\boldsymbol{\Omega}_{,i} - \boldsymbol{M}\cdot\boldsymbol{H} + U_\mathrm{an}. \quad (54)\end{aligned}$$

In order to quantize the Hamiltonian, we use the canonical transformation

$$\boldsymbol{\pi}' = \boldsymbol{\pi} - a''\boldsymbol{M}_\mathrm{o} - 2(a+a'M^2)/a''\boldsymbol{H} \quad (55)$$

which eliminates linear terms of its expansion in the field and momentum components. The quantized boson fields and momenta:

$$\begin{aligned}
\varphi_x &= \frac{\hbar^{1/2}\gamma}{4(V\chi_\perp)^{1/2}}\sum_{\mathbf{k}}\omega_{\perp\mathbf{k}}^{-1/2}[(a_{\mathbf{k}-}+a^\dagger_{-\mathbf{k}-})+(a_{\mathbf{k}+}+b^\dagger_{-\mathbf{k}+})]\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}},\\
\pi'_x &= \mathrm{i}\frac{(\hbar\chi_\perp)^{1/2}}{V^{1/2}\gamma}\sum_{\mathbf{k}}\omega_{\perp\mathbf{k}}^{1/2}[(a^\dagger_{\mathbf{k}-}-a_{-\mathbf{k}-})+(a^\dagger_{\mathbf{k}+}-a_{-\mathbf{k}+})]\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}},\\
\varphi_y &= \mathrm{i}\frac{\hbar^{1/2}\gamma}{4(V\chi_\perp)^{1/2}}\sum_{\mathbf{k}}\omega_{\perp\mathbf{k}}^{-1/2}[(a^\dagger_{-\mathbf{k}-}-a_{\mathbf{k}-})-(a^\dagger_{-\mathbf{k}+}-a_{\mathbf{k}+})]\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}},\\
\pi'_y &= \frac{(\hbar\chi_\perp)^{1/2}}{V^{1/2}\gamma}\sum_{\mathbf{k}}\omega_{\perp\mathbf{k}}^{1/2}[-(a^\dagger_{\mathbf{k}-}+a_{-\mathbf{k}-})+(a^\dagger_{\mathbf{k}+}+a_{-\mathbf{k}+})]\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}},\\
\varphi_z &= \frac{\hbar^{1/2}\gamma}{2(2V\chi_\parallel)^{1/2}}\sum_{\mathbf{k}}\omega_{\parallel\mathbf{k}}^{-1/2}(b_{\mathbf{k}}+b^\dagger_{-\mathbf{k}})\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}},\\
\pi'_z &= \mathrm{i}\frac{(2\hbar\chi_\parallel)^{1/2}}{V^{1/2}\gamma}\sum_{\mathbf{k}}\omega_{\parallel\mathbf{k}}^{1/2}(b^\dagger_{\mathbf{k}}-b_{-\mathbf{k}})\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}},
\end{aligned}\tag{56}$$

satisfy the canonical commutation rules. Here

$$\omega_{\perp\mathbf{k}} = \frac{\gamma}{2}\left[\left(\frac{M}{\chi_\perp}+(1-\xi)H\right)^2+4\xi H^2+\frac{4M}{\gamma\chi_\perp}a_\perp \mathrm{k}^2+\frac{2(\alpha_1+\alpha_2)}{\chi_\perp}\right]^{1/2}, \tag{57}$$

and $a^{(\dagger)}_{\mathbf{k}\pm}$, $b^{(\dagger)}_{\mathbf{k}}$ denote the annihilation (creation) operators for the two shear and one longitudinal magnon modes.

We consider the quantized Hamiltonian up to the fourth order of expansion in the annihilation (creation) operators with the diagonal bilinear part

$$\mathcal{H}_2 = \hbar\sum_{\mathbf{k}}(\omega_{\mathbf{k}-}a^\dagger_{\mathbf{k}-}a_{\mathbf{k}-}+\omega_{\mathbf{k}+}a^\dagger_{\mathbf{k}+}a_{\mathbf{k}+}+\omega_{\parallel\mathbf{k}}b^\dagger_{\mathbf{k}}b_{\mathbf{k}}). \tag{58}$$

For $\boldsymbol{H}=0$, the third-order term takes the form:

$$\begin{aligned}
\mathcal{H}_3 = \sum_{\mathbf{123}} & A_{-+}(\mathbf{123})[(\mathrm{i}b^\dagger_{\mathbf{1}}a_{2-}a_{3+}+\mathrm{h.c.})\Delta(\mathbf{1}-\mathbf{2}-\mathbf{3})+(\mathrm{i}b^\dagger_{\mathbf{1}}a^\dagger_{\mathbf{2}-}a^\dagger_{\mathbf{3}+}+\mathrm{h.c.})\Delta(\mathbf{1}+\mathbf{2}+\mathbf{3})]\\
&+\sum_{\mathbf{123}}A_{--}(\mathbf{123})[(\mathrm{i}b^\dagger_{\mathbf{1}}a^\dagger_{\mathbf{2}-}a_{3-}+\mathrm{h.c.})\Delta(\mathbf{1}+\mathbf{2}-\mathbf{3})\\
&+(\mathrm{i}b^\dagger_{\mathbf{1}}a_{2-}a^\dagger_{\mathbf{3}-}+\mathrm{h.c.})\Delta(\mathbf{1}-\mathbf{2}+\mathbf{3})]\\
&+\sum_{\mathbf{123}}A_{++}(\mathbf{123})[(\mathrm{i}b^\dagger_{\mathbf{1}}a^\dagger_{\mathbf{2}+}a_{3+}+\mathrm{h.c.})\Delta(\mathbf{1}+\mathbf{2}-\mathbf{3})\\
&+(\mathrm{i}b^\dagger_{\mathbf{1}}a_{2+}a^\dagger_{\mathbf{3}+}+\mathrm{h.c.})\Delta(\mathbf{1}-\mathbf{2}+\mathbf{3})]\\
&+\sum_{\mathbf{123}}[B_{\mathrm{I}}(\mathbf{123})(\mathrm{i}b^\dagger_{\mathbf{1}}b_2b_3+\mathrm{h.c.})\Delta(\mathbf{1}-\mathbf{2}-\mathbf{3})\\
&-B_{\mathrm{I}}(\mathbf{312})(\mathrm{i}b^\dagger_{\mathbf{1}}b^\dagger_{\mathbf{2}}b_3+\mathrm{h.c.})\Delta(\mathbf{1}+\mathbf{2}-\mathbf{3})\\
&+B_{\mathrm{II}}(\mathbf{123})(\mathrm{i}b^\dagger_{\mathbf{1}}b^\dagger_{\mathbf{2}}b^\dagger_{\mathbf{3}}+\mathrm{h.c.})\Delta(\mathbf{1}+\mathbf{2}+\mathbf{3})\\
&-B_{\mathrm{I}}(\mathbf{213})(\mathrm{i}b^\dagger_{\mathbf{1}}b_2b^\dagger_{\mathbf{3}}+\mathrm{h.c.})\Delta(\mathbf{1}-\mathbf{2}+\mathbf{3})],
\end{aligned}\tag{59}$$

where

$$
\begin{aligned}
A_{-+}(\mathbf{123}) &= \frac{\hbar^{3/2}}{V^{1/2}}\frac{\gamma}{2^{3/2}\chi_\parallel^{1/2}}\omega_{\parallel\mathbf{1}}^{1/2},\\
A_{--}(\mathbf{123}) &= \frac{\hbar^{3/2}}{V^{1/2}}\frac{\gamma}{2^{5/2}\chi_\parallel^{1/2}}\left[(2-\chi_\parallel/\chi_\perp)\omega_{\parallel\mathbf{1}}^{1/2}+\frac{2\chi_\perp}{\chi_\parallel}\frac{a_\perp|\mathbf{1}|^2}{\omega_{\parallel\mathbf{1}}^{1/2}}\right],\\
A_{++}(\mathbf{123}) &= \frac{\hbar^{3/2}}{V^{1/2}}\frac{\gamma}{2^{5/2}\chi_\parallel^{1/2}}\left(\frac{\chi_\parallel}{\chi_\perp}\omega_{\parallel\mathbf{1}}^{1/2}-\frac{2\chi_\perp}{\chi_\parallel}\frac{a_\perp|\mathbf{1}|^2}{\omega_{\parallel\mathbf{1}}^{1/2}}\right),
\end{aligned}
\tag{60}
$$

and $B_{\mathrm{I(II)}}(\mathbf{123})$ are similar as in (21). The part of $\mathcal{H}_4$ important for evaluation of the low-energy shear-mode damping coefficient (corresponding to processes conserving the number of magnons) is

$$
\begin{aligned}
\mathcal{H}_{4C} = \sum_{\mathbf{1234}} C(\mathbf{123k})[&(a_{\mathbf{1}-}^\dagger a_{\mathbf{2}-}a_{\mathbf{3}-}a_{\mathbf{4}-}^\dagger+\mathrm{h.c.})\Delta(\mathbf{1}-\mathbf{2}-\mathbf{3}+\mathbf{4})\\
&+(a_{\mathbf{1}-}^\dagger a_{\mathbf{2}-}a_{\mathbf{3}-}^\dagger a_{\mathbf{4}-}+\mathrm{h.c.})\Delta(\mathbf{1}-\mathbf{2}+\mathbf{3}-\mathbf{4})].
\end{aligned}
\tag{61}
$$

The amplitude

$$
C(\mathbf{1234}) = \frac{\hbar^2}{V}\frac{\chi_\parallel c_\parallel^2}{32M^2}(\mathbf{1}+\mathbf{2})\cdot(\mathbf{3}+\mathbf{4}) \tag{62}
$$

is essentially different from the corresponding amplitude for RCF because it is of higher order in $T/\frac{\hbar\gamma M}{\chi_\parallel}$, considering the interaction of thermal magnons (with energies of the order of $T$).

Writing the density of the magnetic moment (5) in canonical variables:

$$
\boldsymbol{m} = \frac{\gamma}{2}\{(\boldsymbol{\pi}'-\boldsymbol{\eta}')+\boldsymbol{\varphi}\times(\boldsymbol{\pi}'-\boldsymbol{\eta}')+[\boldsymbol{\varphi}\cdot(\boldsymbol{\pi}'-\boldsymbol{\eta}')]\boldsymbol{\varphi}\}+\boldsymbol{M}+\hat{\chi}\boldsymbol{H}, \tag{63}
$$

where $\boldsymbol{\eta}' = (a''+2a'\boldsymbol{M}\cdot\boldsymbol{H}/a'')(\boldsymbol{\mu}-\boldsymbol{M}_\mathrm{o})+2a/a''(\boldsymbol{\nu}-\boldsymbol{H})$, we perform its quantization.

The regime of parameters important for comparison of the results of calculations with the results for RCF:

$$
\omega_{\perp\mathbf{k}} \simeq \frac{\gamma M}{2\chi_\perp} \gg \frac{\gamma}{2}\left[\frac{4M}{\gamma\chi_\perp}a_\perp \mathrm{k}^2+\frac{2(\alpha_1+\alpha_2)}{\chi_\perp}\right]^{1/2} \tag{64}
$$

relates to the strongly magnetized system. Performing estimations similar as for RCF one finds the small parameters for the calculations $T/\theta_\mathrm{c}$, $\hbar\omega_{\mathbf{k}-}/\theta_\mathrm{c}$, $\hbar\omega_{\parallel\mathbf{k}}/\theta_\mathrm{c}$, $T/\theta_\mathrm{o}$, $\hbar\omega_{\mathbf{k}-}/\theta_\mathrm{o}$, $\hbar\omega_{\parallel\mathbf{k}}/\theta_\mathrm{o}$, $T/\frac{\hbar\gamma M}{\chi_\parallel}$, $\omega_{\mathbf{k}-}/\frac{\gamma M}{\chi_\parallel}$ and $\omega_{\parallel\mathbf{k}}/\frac{\gamma M}{\chi_\parallel}$. In this regime the high energy shear magnons do not thermalize since $\hbar\omega_{\mathbf{k}+} \simeq 2\hbar\omega_{\perp\mathbf{k}} \simeq \hbar\gamma M/\chi_\perp \gg T$. The opposite case $M = 0$, $\chi_\parallel = \chi_\perp$ corresponds to the GSG phase which has been studied in [12].

### *4.2. Linear response functions*

As done for RCF, we carry out the calculations of the dynamical response Green functions in the vicinity of the spin-wave resonance expanding the components of the magnetization operator to the third order in the magnon creation (annihilation) operators. Their singular parts may be written using the one-particle Green functions $G_{\alpha\beta}(\mathbf{k},\omega) = \langle\langle b_{\mathbf{k}\alpha}(t), b_{\mathbf{k}\beta}^\dagger(0)\rangle\rangle_\omega$, ($\boldsymbol{b}_\mathbf{k} = (a_{\mathbf{k}-}, a_{\mathbf{k}+}, b_\mathbf{k}, a_{-\mathbf{k}-}^\dagger, a_{-\mathbf{k}+}^\dagger, b_{-\mathbf{k}}^\dagger)$). The relevant diagonal functions:

$$
\begin{aligned}
&G_{\mu\mu}(\mathbf{k},\omega) \simeq 1/[\omega-\omega_{\mathbf{k}\mu}-\Sigma_\mu(\mathbf{k},\omega)],\\
&G_{\mu+3,\mu+3}(\mathbf{k},\omega) = G_{\mu\mu}^*(-\mathbf{k},-\omega) \simeq -1/[\omega+\omega_{\mathbf{k}\mu}+\Sigma_{\mu\mu}^*(-\mathbf{k},-\omega)]
\end{aligned}
\tag{65}
$$

($\mu = 1, 2, 3$, $\omega_{\mathbf{k}1} = \omega_{\mathbf{k}-}$, $\omega_{\mathbf{k}2} = \omega_{\mathbf{k}+}$, $\omega_{\mathbf{k}3} = \omega_{\parallel\mathbf{k}}$) are of lower order in $\hat{\Sigma}(\mathbf{k},\omega)/\omega$ (which is small in the vicinity of the resonance) than irrelevant off-diagonal functions.

Decomposing all the one-particle function terms by using (A.7), we find the singular part of the susceptibility:

$$\chi_{xx}(\mathbf{k},\omega) = -\frac{\gamma^2}{4\hbar}\Bigg(\frac{\hbar\chi_\perp}{\gamma^2\omega_{\perp\mathbf{k}}}\{[\omega_{\perp\mathbf{k}} + (\gamma M/2\chi_\perp)]^2[G_{11}(\mathbf{k},\omega) + G^*_{11}(-\mathbf{k},-\omega)]$$
$$+ [\omega_{\perp\mathbf{k}} - (\gamma M/2\chi_\perp)]^2[G_{22}(\mathbf{k},\omega) + G^*_{22}(-\mathbf{k},-\omega)]\}$$
$$+ 2\mathrm{i}\frac{\hbar^{1/2}\chi_\perp^{1/2}}{\gamma^2\omega_{\perp\mathbf{k}}^{1/2}}[\Lambda^{\mu}_{(\pi_z\varphi_y-\varphi_z\pi_y+4M/\gamma\varphi_z\varphi_x)^\dagger}(\mathbf{k},\omega) + \Lambda^{\mu}_{[(\varphi\cdot\pi)\varphi_x]^\dagger}(\mathbf{k},\omega)]$$
$$\times\{-[\omega_{\perp\mathbf{k}} + (\gamma M/2\chi_\perp)]\delta_{1\mu}G_{11}(\mathbf{k},\omega) - [\omega_{\perp\mathbf{k}} - (\gamma M/2\chi_\perp)]\delta_{2\mu}G_{22}(\mathbf{k},\omega)\}$$
$$+ 2\mathrm{i}\frac{\hbar^{1/2}\chi_\perp^{1/2}}{\gamma^2\omega_{\perp\mathbf{k}}^{1/2}}[\Lambda^{\mu*}_{(\pi_z\varphi_y-\varphi_z\pi_y+4M/\gamma\varphi_z\varphi_x)^\dagger}(-\mathbf{k},-\omega) + \Lambda^{\mu*}_{[(\varphi\cdot\pi)\varphi_x]^\dagger}(-\mathbf{k},-\omega)]$$
$$\times\{[\omega_{\perp\mathbf{k}} + (\gamma M/2\chi_\perp)]\delta_{1\mu}G^*_{11}(-\mathbf{k},-\omega)$$
$$+ [\omega_{\perp\mathbf{k}} - (\gamma M/2\chi_\perp)]\delta_{2\mu}G^*_{22}(-\mathbf{k},-\omega)\} + W_{xx}(\mathbf{k},\omega)\Bigg), \tag{66a}$$

$$\chi_{yy}(\mathbf{k},\omega) = -\frac{\gamma^2}{4\hbar}\Bigg(\frac{\hbar\chi_\perp}{\gamma^2\omega_{\perp\mathbf{k}}}\{[\omega_{\perp\mathbf{k}} + (\gamma M/2\chi_\perp)]^2[G_{11}(\mathbf{k},\omega) + G^*_{11}(-\mathbf{k},-\omega)]$$
$$+ [\omega_{\perp\mathbf{k}} - (\gamma M/2\chi_\perp)]^2[G_{22}(\mathbf{k},\omega) + G^*_{22}(-\mathbf{k},-\omega)]\}$$
$$+ 2\frac{\hbar^{1/2}\chi_\perp^{1/2}}{\gamma^2\omega_{\perp\mathbf{k}}^{1/2}}[\Lambda^{\mu}_{(-\pi_z\varphi_x+\varphi_z\pi_x+4M/\gamma\varphi_z\varphi_y)^\dagger}(\mathbf{k},\omega) + \Lambda^{\mu}_{[(\varphi\cdot\pi)\varphi_y]^\dagger}(\mathbf{k},\omega)]$$
$$\times\{-[\omega_{\perp\mathbf{k}} + (\gamma M/2\chi_\perp)]\delta_{1\mu}G_{11}(\mathbf{k},\omega) + [\omega_{\perp\mathbf{k}} - (\gamma M/2\chi_\perp)]\delta_{2\mu}G_{22}(\mathbf{k},\omega)\}$$
$$+ 2\frac{\hbar^{1/2}\chi_\perp^{1/2}}{\gamma^2\omega_{\perp\mathbf{k}}^{1/2}}[\Lambda^{\mu*}_{(-\pi_z\varphi_x+\varphi_z\pi_x+4M/\gamma\varphi_z\varphi_y)^\dagger}(-\mathbf{k},-\omega) + \Lambda^{\mu*}_{[(\varphi\cdot\pi)\varphi_y]^\dagger}(-\mathbf{k},-\omega)]$$
$$\times\{-[\omega_{\perp\mathbf{k}} + (\gamma M/2\chi_\perp)]\delta_{1\mu}G^*_{11}(-\mathbf{k},-\omega)$$
$$+ [\omega_{\perp\mathbf{k}} - (\gamma M/2\chi_\perp)]\delta_{2\mu}G^*_{22}(-\mathbf{k},-\omega)\} + W_{yy}(\mathbf{k},\omega)\Bigg), \tag{66b}$$

$$\chi_{xy}(\mathbf{k},\omega) = \chi^*_{yx}(\mathbf{k},-\omega) = -\frac{\gamma^2}{4\hbar}\Bigg(\mathrm{i}\frac{\hbar\chi_\perp}{\gamma^2\omega_{\perp\mathbf{k}}}\{[\omega_{\perp\mathbf{k}} + (\gamma M/2\chi_\perp)]^2[G_{11}(\mathbf{k},\omega) - G^*_{11}(-\mathbf{k},-\omega)]$$
$$+ [\omega_{\perp\mathbf{k}} - (\gamma M/2\chi_\perp)]^2[-G_{22}(\mathbf{k},\omega) + G^*_{22}(-\mathbf{k},-\omega)]\}$$
$$+ \frac{\hbar^{1/2}\chi_\perp^{1/2}}{\gamma^2\omega_{\perp\mathbf{k}}^{1/2}}[-\mathrm{i}\Lambda^{1}_{(\varphi_z\pi_x-\pi_z\varphi_x+4M/\gamma\varphi_z\varphi_y)^\dagger}(\mathbf{k},\omega) - \Lambda^{1}_{(-\varphi_z\pi_y+\pi_z\varphi_y+4M/\gamma\varphi_z\varphi_x)^\dagger}(\mathbf{k},\omega)$$
$$- \mathrm{i}\Lambda^{1}_{[(\varphi\cdot\pi)\varphi_y]^\dagger}(\mathbf{k},\omega) - \Lambda^{1}_{[(\varphi\cdot\pi)\varphi_x]^\dagger}(\mathbf{k},\omega)][\omega_{\perp\mathbf{k}} + (\gamma M/2\chi_\perp)]G_{11}(\mathbf{k},\omega)$$
$$+ \frac{\hbar^{1/2}\chi_\perp^{1/2}}{\gamma^2\omega_{\perp\mathbf{k}}^{1/2}}[\mathrm{i}\Lambda^{1*}_{(\varphi_z\pi_x-\pi_z\varphi_x+4M/\gamma\varphi_z\varphi_y)^\dagger}(-\mathbf{k},-\omega)$$
$$- \Lambda^{1*}_{(-\varphi_z\pi_y+\pi_z\varphi_y+4M/\gamma\varphi_z\varphi_x)^\dagger}(-\mathbf{k},-\omega)$$
$$+ \mathrm{i}\Lambda^{1*}_{[(\varphi\cdot\pi)\varphi_y]^\dagger}(-\mathbf{k},-\omega) - \Lambda^{1*}_{[(\varphi\cdot\pi)\varphi_x]^\dagger}(-\mathbf{k},-\omega)]$$
$$\times[\omega_{\perp\mathbf{k}} + (\gamma M/2\chi_\perp)]G^*_{11}(-\mathbf{k},-\omega)$$
$$+ \frac{\hbar^{1/2}\chi_\perp^{1/2}}{\gamma^2\omega_{\perp\mathbf{k}}^{1/2}}[-\mathrm{i}\Lambda^{2}_{(\varphi_z\pi_x-\pi_z\varphi_x+4M/\gamma\varphi_z\varphi_y)^\dagger}(\mathbf{k},\omega) + \Lambda^{2}_{(-\varphi_z\pi_y+\pi_z\varphi_y+4M/\gamma\varphi_z\varphi_x)^\dagger}(\mathbf{k},\omega)$$
$$- \mathrm{i}\Lambda^{2}_{[(\varphi\cdot\pi)\varphi_y]^\dagger}(\mathbf{k},\omega) + \Lambda^{2}_{[(\varphi\cdot\pi)\varphi_x]^\dagger}(\mathbf{k},\omega)][\omega_{\perp\mathbf{k}} - (\gamma M/2\chi_\perp)]G_{22}(\mathbf{k},\omega)$$

$$+ \frac{\hbar^{1/2}\chi_\perp^{1/2}}{\gamma^2\omega_{\perp\mathbf{k}}^{1/2}}[\mathrm{i}\Lambda^{2*}_{(\varphi_z\pi_x-\pi_z\varphi_x+4M/\gamma\varphi_z\varphi_y)^\dagger}(-\mathbf{k},-\omega)$$
$$+ \Lambda^{2*}_{(-\varphi_z\pi_y+\pi_z\varphi_y+4M/\gamma\varphi_z\varphi_x)^\dagger}(-\mathbf{k},-\omega) + \mathrm{i}\Lambda^{2*}_{[(\varphi\cdot\pi)\varphi_y]^\dagger}(-\mathbf{k},-\omega)$$
$$+ \Lambda^{2*}_{[(\varphi\cdot\pi)\varphi_x]^\dagger}(-\mathbf{k},-\omega)][\omega_{\perp\mathbf{k}} - (\gamma M/2\chi_\perp)]G^*_{22}(-\mathbf{k},-\omega) + W_{xy}(\mathbf{k},\omega)\Bigg), \tag{66c}$$

$$\chi_{zz}(\mathbf{k},\omega) = -\frac{\gamma^2}{2\hbar}\Bigg\{\frac{\hbar\chi_\parallel}{\gamma^2}[G_{33}(\mathbf{k},\omega) + G^*_{33}(-\mathbf{k},-\omega)]$$
$$- \mathrm{i}\frac{(2\hbar\chi_\parallel)^{1/2}}{\gamma}\omega_{\parallel\mathbf{k}}^{1/2}[\Lambda^3_{(2M/\gamma\varphi_z^2)^\dagger}(\mathbf{k},\omega) + \Lambda^3_{[\pi_y\varphi_x-\varphi_y\pi_x-2M/\gamma(\varphi_x^2+\varphi_y^2)]^\dagger}(\mathbf{k},\omega)$$
$$+ \Lambda^3_{[(\varphi\cdot\pi)\varphi_x]^\dagger}(\mathbf{k},\omega)]G_{33}(\mathbf{k},\omega)$$
$$+ \mathrm{i}\frac{(2\hbar\chi_\parallel)^{1/2}}{\gamma}\omega_{\parallel\mathbf{k}}^{1/2}[\Lambda^{3*}_{(2M/\gamma\varphi_z^2)^\dagger}(-\mathbf{k},-\omega) + \Lambda^{3*}_{[\pi_y\varphi_x-\varphi_y\pi_x-2M/\gamma(\varphi_x^2+\varphi_y^2)]^\dagger}(-\mathbf{k},-\omega)$$
$$+ \Lambda^{3*}_{[(\varphi\cdot\pi)\varphi_x]^\dagger}(-\mathbf{k},-\omega)]G^*_{33}(-\mathbf{k},-\omega) + W_{zz}(\mathbf{k},\omega)\Bigg\}, \tag{66d}$$

$$\chi_{xz} = \chi_{zx} = \chi_{yz} = \chi_{zy} = 0. \tag{66e}$$

The functions $W_{ij}(\mathbf{k},\omega)$ contain singular contributions to many-particle functions which are of higher order in $T/\theta_c$, $\hbar\omega_{\mathbf{k}-}/\theta_c, \ldots$, than the contributions from the one-particle functions. Calculating the functions $\Lambda^\alpha_A(\mathbf{k},\omega)$ of (66*a*)–(66*e*), we obtain the susceptibility components:

$$\chi_{xx}(\mathbf{k},\omega) = \chi_{yy}(\mathbf{k},\omega) = -\frac{\chi_\perp}{4\omega_{\perp\mathbf{k}}}\{[\omega_{\perp\mathbf{k}} + (\gamma M/2\chi_\perp)]^2[1 + (T/\theta_c)^{3/2}\mathrm{e}^{-\hbar\gamma M/\chi_\perp T}]$$
$$\times [(\omega - \omega_{\mathbf{k}-} + \mathrm{i}\Gamma_{\mathbf{k}-})^{-1} - (\omega + \omega_{\mathbf{k}-} + \mathrm{i}\Gamma_{\mathbf{k}-})^{-1}]$$
$$+ [\omega_{\perp\mathbf{k}} - (\gamma M/2\chi_\perp)]^2[1 + (T/\theta_c)^{3/2}]$$
$$\times [(\omega - \omega_{\mathbf{k}+} + \mathrm{i}\Gamma_{\mathbf{k}+})^{-1} - (\omega + \omega_{\mathbf{k}+} + \mathrm{i}\Gamma_{\mathbf{k}+})^{-1}]\}, \tag{67a}$$

$$\chi_{xy}(\mathbf{k},\omega) = \chi^*_{yx}(\mathbf{k},-\omega) = -\frac{\mathrm{i}\chi_\perp}{4\omega_{\perp\mathbf{k}}}\{[\omega_{\perp\mathbf{k}} + (\gamma M/2\chi_\perp)]^2[1 - (T/\theta_c)^{3/2}\mathrm{e}^{-\hbar\gamma M/\chi_\perp T}]$$
$$\times [(\omega - \omega_{\mathbf{k}-} + \mathrm{i}\Gamma_{\mathbf{k}-})^{-1} + (\omega + \omega_{\mathbf{k}-} + \mathrm{i}\Gamma_{\mathbf{k}-})^{-1}]$$
$$- [\omega_{\perp\mathbf{k}} - (\gamma M/2\chi_\perp)]^2[1 - (T/\theta_c)^{3/2}]$$
$$\times [(\omega - \omega_{\mathbf{k}+} + \mathrm{i}\Gamma_{\mathbf{k}+})^{-1} + (\omega + \omega_{\mathbf{k}+} + \mathrm{i}\Gamma_{\mathbf{k}+})^{-1}]\}, \tag{67b}$$

$$\chi_{zz}(\mathbf{k},\omega) = -\frac{\chi_\parallel}{2}[1 + (T/\theta_o)^2]^2\omega_{\parallel\mathbf{k}}[(\omega - \omega_{\parallel\mathbf{k}} + \mathrm{i}\Gamma_{\parallel\mathbf{k}})^{-1} - (\omega + \omega_{\parallel\mathbf{k}} + \mathrm{i}\Gamma_{\parallel\mathbf{k}})^{-1}]. \tag{67c}$$

Here $\Gamma_{\mathbf{k}-} = -\,\mathrm{Im}\,\Sigma_{11}(\mathbf{k},\omega_{\mathbf{k}-})$, $\Gamma_{\mathbf{k}+} = -\,\mathrm{Im}\,\Sigma_{22}(\mathbf{k},\omega_{\mathbf{k}+})$ and $\Gamma_{\parallel\mathbf{k}} = -\,\mathrm{Im}\,\Sigma_{33}(\mathbf{k},\omega_{\parallel\mathbf{k}})$.

The diagonalized susceptibility is of the form (37) and their poles are the same as the poles of the one-particle functions. The temperature dependence of $\hat{\chi}(\mathbf{k},\omega)$ is essentially different from the corresponding dependence for RCF since the exponential factors $\mathrm{e}^{-\hbar\gamma M/\chi_\perp T}$ in the numerators of its shear components are very small.

Let us evaluate the contributions to the spin-wave damping coefficients of the low-energy shear mode arising from $\mathcal{H}_3$ (denoted by $\Gamma^3_{\mathbf{k}-}$) and from $\mathcal{H}_4$ (denoted by $\Gamma^4_{\mathbf{k}-}$) for $\alpha_1, \ldots, \alpha_4 \approx 0$. The additional index (2) at the top of the symbols denotes the contributions of the second order of the interaction. Evaluating the expressions (C.1) and (C.2) from appendix C, [24], we find that $\Gamma^{3(2)}_{\mathbf{k}-} \propto \exp(-\hbar c_\parallel^2/a_\perp T)$; this is small (negligible) compared to

$$\Gamma^{4(2)}_{\mathbf{k}-} \sim \begin{cases} 10^{-1}(\chi_\parallel/\chi_\perp)^2\left(\dfrac{c_\parallel^2}{a_\perp}\Big/\dfrac{\gamma M}{\chi_\perp}\right)^2 & \\ \quad\times\, a_\perp \mathrm{k}^2(\hbar a_\perp \mathrm{k}^2/\theta_c)(T/\theta_c)^2 & \hbar a_\perp \mathrm{k}^2, \hbar c_\parallel \mathrm{k} \ll T \\ 10^{-1}(\chi_\parallel/\chi_\perp)^2\left(\dfrac{c_\parallel^2}{a_\perp}\Big/\dfrac{\gamma M}{\chi_\perp}\right)^2 & \\ \quad\times\, a_\perp \mathrm{k}^2(\hbar a_\perp \mathrm{k}^2/\theta_c)^{3/2}(T/\theta_c)^{3/2} & T \ll \hbar a_\perp \mathrm{k}^2, \hbar c_\parallel \mathrm{k}. \end{cases} \tag{68}$$

Finding a more accurate form of the damping coefficient for the regime $\hbar\gamma(\alpha_1+\alpha_2)/(2M) \ll \hbar\omega_{\mathbf{k}-} \ll T$:

$$\Gamma^{4(2)}_{\mathbf{k}-} = \omega_{\mathbf{k}-}(\hbar a_\perp \mathrm{k}^2/\theta_c)(T/\theta_c)^2[C + C'\ln(T/\hbar a_\perp \mathrm{k}^2) + C''\ln^2(T/\hbar a_\perp \mathrm{k}^2)] \tag{69}$$

($C, C', C''$ are constants), we have used the total scattering amplitude in the expression for $\Gamma^{4(2)}_{\mathbf{k}-}$ (C.2) written in the form

$$M_-(\mathbf{p}, \mathbf{s}, \mathbf{k}+\mathbf{p}-\mathbf{s}, \mathbf{k}) \propto |\mathbf{p}+\mathbf{k}|^2 + 4\mathbf{k}\cdot\mathbf{p} \tag{70}$$

on the mass surface defined by the equation $\omega_{\mathbf{k}-} + \omega_{\mathbf{p}-} - \omega_{\mathbf{s}-} - \omega_{(\mathbf{k}+\mathbf{p}-\mathbf{s})-} = 0$.

Accidentally, the wavevector dependence of $\Gamma^{4(2)}_{\mathbf{k}-}$ is similar to that of the Heisenberg ferromagnet, up to multiplicative constants [20, 24, 26, 27], but it is essentially different from the suitable one for the shear mode of RCF, being a more similar system to SSG in terms of the magnetic anisotropy. This is because of the lack of a constant term in (70) that was present in (41). The dependence $\Gamma_{\mathbf{k}-} \propto \mathrm{k}^4$ has been observed for disordered $Ni_{1-x}Mn_x$ ($x \approx 0.13$), which is a SSG [36]. In this system the alignment of a part of the Mn spins is antiparallel to the spontaneous moment $M$, [37]. Unfortunately, there are no available data on spin-wave damping in semiconducting FSGs to the author's knowledge, which could be interesting since ferromagnetic (RCF) as well as ferrimagnetic (SSG) resonances have been observed for thick (2 $\mu$m) films of $Ga_{1-x}Mn_xAs$, dependent on the magnetic ion concentration $x$ [38], not for the bulk material, however, for technological reasons [39]. Recent calculations for this compound suggest the existence of both types of FSG ordering [40].

## 5. Summary and outlook

We have constructed the quantum description of the spin-wave excitations in RCF and SSG in the framework of a phenomenological approach which is general and applicable to a wide class of materials since it is independent of the mechanism of spin interaction (based only on their symmetry). It should be mentioned, however, that the macroscopic description is restricted to systems of large magnetic moments since it treats spin as a vector, which makes it inapplicable to itinerant FSGs where a part of the spins is very small [41], resulting in the experimentally observed low-temperature disappearance of spin waves [42]. The problem of quantization of the macroscopic excitations has been solved for the RCF model, which is non-trivial due to its constrained dynamics.

We have found differences in the dynamical susceptibility of RCF compared to that of the Heisenberg ferromagnet. Especially important differences in the shear spin-wave coefficients of damping due to the magnon–magnon interaction (the appearance of the dispersion gap in the long-wavelength limit) have been interpreted as a consequence of the presence of the anisotropy in the FSG's Hamiltonian which is an essential topic in the description of the dynamics of partially ordered or non-collinear structures [14]. The differences found in the dynamical susceptibility of RCF compared to SSG enable one to distinguish both phases

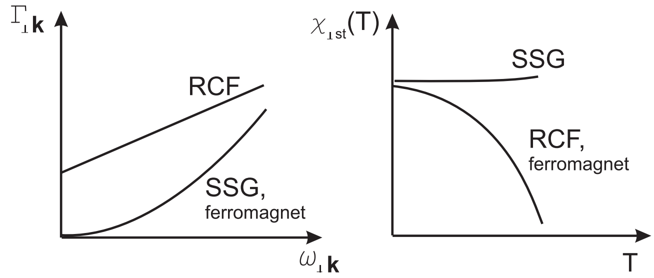


**Figure 2.** Dispersion of damping coefficients of the long-wavelength shear spin waves for RCF, SSG and ferromagnet, next to the temperature dependence of the magnon contribution to the shear static susceptibility of these systems, ($\chi_\perp(T)$ for the ferromagnet from [27]).

experimentally. There are different dispersions of the damping coefficients of spin waves of both phases (figure 2), but also temperature dependences of the contributions to the dynamical and to the easily measurable static susceptibility are different (the static one may be found from the dynamical one in the limit $\omega = 0, \mathbf{k} \to 0$). The cross section for the neutron scattering from RCF via spin waves has been evaluated in order to make possible the precise interpretation of the spectroscopy observations.

Let us notice, however, that the magnon–magnon channel of spin-wave relaxation, which is a non-dissipative channel, is expected to be the most important for the stiff shear modes of FSG excitations while the longitudinal mode is usually soft in real magnets due to the presence of inhomogeneity (a mictomagnetism) [43]. The inclusion of inhomogeneity may be easily carried out via treating the vector forms $\mathbf{a} = \dot{\boldsymbol{\Omega}}(\varphi)$, $\mathbf{b}_i = \boldsymbol{\Omega}_{,i}(\psi)$ as components of a Yang–Mills field for the transformation $SO(3)$ of the spin vector, as was proposed for the GSG by Dzyaloshinskii and Volovik for a different problem (the so-called 'frustration') [44]. Here $\varphi$ and $\psi$ denote the parameters of rotation of the spin (isotopic) space from the temporal equilibrium state and from the initial state, respectively. The general method of construction of the relaxation function developed in application to homogeneous disordered magnets in [45] may be used to this case, leading to the appearance of two longitudinal over-damped modes instead of one propagative longitudinal mode, according to the assumption of [46] and according to numerous experimental observations of an additional magnetic central peak in neutron scattering intensity.

The channel of magnon relaxation via elastic scattering from acoustical phonons is generally important due to the comparable energies of magnons and phonons. We have studied this in the framework of a macroscopic theory using the method of [22, 47]. The application of this method to the spin glass was presented in detail in [48]. Expanding the Hamiltonian in the small deformational parameters $\hat{u}$, $\boldsymbol{\Omega}_{,i} \cdot \boldsymbol{u}_{,j}$, $\boldsymbol{M} \cdot \boldsymbol{u}_{,i}$, $\hat{v}$ ($v_{ij} = 1/2(u_{i,k}O_{kj} + O_{ik}u_{k,j} + u_{i,k}O_{kl}u_{l,j})$), where $\boldsymbol{u}$ denotes the deformation field and $\hat{u}$ the deformation tensor, one may find via the canonical quantization the shear magnon–phonon interaction Hamiltonian for RCF being of the form

$$\mathcal{H}_{\text{s-ph}} = \sum_{\mathbf{123}\nu} A_{\text{s-ph}}(\mathbf{123}\nu)[(\mathrm{i}c^\dagger_{\mathbf{1}\nu} a^\dagger_{\mathbf{2}} a_{\mathbf{3}} + \text{h.c.})\Delta(\mathbf{1}+\mathbf{2}-\mathbf{3}) + (\mathrm{i}c^\dagger_{\mathbf{1}\nu} a_{\mathbf{2}} a^\dagger_{\mathbf{3}} + \text{h.c.})\Delta(\mathbf{1}-\mathbf{2}+\mathbf{3})] \tag{71}$$

with

$$A_{\text{s-ph}}(\mathbf{123}\nu) = \frac{\hbar^{3/2}}{V^{1/2}} \frac{1}{2(2\rho)^{1/2}\Theta} (\omega^{\text{ph}}_{\mathbf{1}\nu})^{-1/2} \{-2\lambda_2(\mathbf{2}\cdot\mathbf{3})(\mathbf{1}\cdot \boldsymbol{e}_{\mathbf{1}\nu}) - \lambda_1[(\mathbf{1}\cdot\mathbf{2})(\mathbf{3}\cdot \boldsymbol{e}_{\mathbf{1}\nu}) + (\mathbf{1}\cdot\mathbf{3})(\mathbf{2}\cdot \boldsymbol{e}_{\mathbf{1}\nu})] + (\beta_1+\beta_2)(\mathbf{1}\cdot \boldsymbol{e}_{\mathbf{1}\nu})\}. \tag{72}$$

Here $c^{(\dagger)}_{\mathbf{k}\nu}$ denotes the annihilation (creation) operator for the acoustical phonon of the mode numbered by $\nu$ ($\nu = 1, 2, 3$) and $\boldsymbol{e}_{\mathbf{k}\nu}$ denotes the corresponding phonon polarization vector, where $\rho$ is the density of mass. In the expression, the terms depending on the constants $\lambda_{1(2)}$ are of exchange origin while those depending on $\beta_{1(2)}$ are of one-ion anisotropy origin. Such a form of the magnon–phonon interaction is identical to that predicted for the Heisenberg ferromagnet, which enables us to exclude qualitative differences in magnon–phonon corrections to the magnon damping coefficient of RCF and ferromagnet [26].

The studies on the magneto-elastic waves which have been carried out for a FSG in [13] give additional information on the long-wavelength spin-wave relaxation in the vicinity of homogeneous resonance (using ultrasound methods).

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## Appendix A. Perturbation method for one-particle functions

The most important terms in the linear-response functions are one-particle functions of the form $\langle\langle b_{\mathbf{k}\alpha}(t), A(0)\rangle\rangle$, where $b_{\mathbf{k}\alpha}$ are the elements of the four-component vector of the creation and annihilation operators in the Heisenberg picture for RCF:

$$\boldsymbol{b}_{\mathbf{k}}(t) = (a_{\mathbf{k}}(t), b_{\mathbf{k}}(t), a^{\dagger}_{-\mathbf{k}}(t), b^{\dagger}_{-\mathbf{k}}(t)), \tag{A.1}$$

or of the six-component vector for SSG, respectively. Let us define the projection operator $PA = \mathrm{i}\langle\langle A, b^{\dagger}_{\mathbf{k}\alpha}\rangle\rangle R^{-1}_{\alpha\beta} b_{\mathbf{k}\beta}$, $(R_{\alpha\beta} = \langle[b_{\mathbf{k}\alpha} b^{\dagger}_{\mathbf{k}\beta}]\rangle)$. The solutions of the Liouville equation for the one-particle operators separate into a linear part and a non-linear perturbation correction:

$$\boldsymbol{b}_{\mathbf{k}}(t) = P\boldsymbol{b}_{\mathbf{k}}(t) + (1-P)\boldsymbol{b}_{\mathbf{k}}(t) = \mathrm{i}\hat{G}(\mathbf{k}, t)\hat{R}^{-1}\boldsymbol{b}_{\mathbf{k}} + \mathrm{i}\int_0^t \hat{G}(\mathbf{k}, s)\hat{R}^{-1}\boldsymbol{f}_{\mathbf{k}}(t-s)\,\mathrm{d}s, \tag{A.2}$$

where $\boldsymbol{f}_{\mathbf{k}}(t) = \mathrm{i}/\hbar \mathrm{e}^{(\mathrm{i}/\hbar(1-P)Lt)}(1-P)L\boldsymbol{b}_{\mathbf{k}}$ and $L$ denotes the Liouville operator $LA = [\mathcal{H}_{\mathrm{o}} + \mathcal{H}_{\mathrm{int}}, A]$ $(L = L_{\mathrm{o}} + L_{\mathrm{int}})$. Using (A.2), one solves the equations of motion for the one-particle functions and finds

$$\hat{G}(\mathbf{k}, \omega) = [(\omega - \hat{\omega}_{\mathbf{k}})\hat{R}^{-1} - \hat{\Sigma}(\mathbf{k}, \omega)]^{-1}, \tag{A.3}$$

where

$$\omega_{\mathbf{k}\alpha\beta} = -\mathrm{i}/\hbar\langle\langle L_{\mathrm{o}} b_{\mathbf{k}\alpha}, b^{\dagger}_{\mathbf{k}\gamma}\rangle\rangle R^{-1}_{\gamma\beta}, \tag{A.4}$$

$$\Sigma_{\alpha\beta}(\mathbf{k}, \omega) = -\mathrm{i}/\hbar\langle\langle L_{\mathrm{int}} b_{\mathbf{k}\alpha}, b^{\dagger}_{\mathbf{k}\gamma}\rangle\rangle R^{-1}_{\gamma\beta} + \langle\langle f_{\mathbf{k}\alpha}(t), f^{\dagger}_{\mathbf{k}\gamma}\rangle\rangle_{\omega} R^{-1}_{\gamma\beta}. \tag{A.5}$$

In order to evaluate the real and imaginary parts of the mass operator, we take into account the second interaction approximation for the expression (A.5):

$$\Sigma_{\alpha\beta}(\mathbf{k}, \omega) = -1/\hbar\langle[L_{\mathrm{int}} b_{\mathbf{k}\alpha}, b^{\dagger}_{\mathbf{k}\gamma}]\rangle_{\mathrm{o}} R^{-1}_{\gamma\beta} - \mathrm{i}/\hbar^2 \int_{-\infty}^{\infty} \mathrm{d}t\, \theta(t)\mathrm{e}^{\mathrm{i}\omega t}$$
$$\times \langle[\mathrm{e}^{\mathrm{i}/\hbar L_{\mathrm{o}} t}(1-P)L_{\mathrm{int}} b_{\mathbf{k}\alpha}, (1-P)L_{\mathrm{int}} b^{\dagger}_{\mathbf{k}\gamma}]\rangle_{\mathrm{o}} R^{-1}_{\gamma\beta}. \tag{A.6}$$

From the decomposition (A.2) the Fourier transform of any one-particle function $\langle\langle b_{\mathbf{k}\alpha}(t), A(0)\rangle\rangle$ can be written as a component of the vector

$$\langle\langle \boldsymbol{b}_{\mathbf{k}}(t), A(0)\rangle\rangle_{\omega} = \hat{G}(\mathbf{k}, \omega)\boldsymbol{\Lambda}_A(\mathbf{k}, \omega), \tag{A.7}$$

where

$$\Lambda_A(\mathbf{k},\omega) = \mathrm{i}\hat{R}^{-1}\langle\langle \boldsymbol{b}_{\mathbf{k}}, A(0)\rangle\rangle + \mathrm{i}\hat{R}^{-1}\langle\langle \boldsymbol{f}_{\mathbf{k}}(t), A(0)\rangle\rangle_\omega. \tag{A.8}$$

The components of (A.8) are calculated up to the first order in interaction:

$$\Lambda^{\alpha}_{A}(\mathbf{k},\omega) = R^{-1}_{\alpha\beta}\langle[b_{\mathbf{k}\beta}, A(0)]\rangle_{\mathrm{o}} + \mathrm{i}/\hbar R^{-1}_{\alpha\beta}\int_{-\infty}^{\infty}\mathrm{d}t\,\theta(t)\mathrm{e}^{\mathrm{i}\omega t}\langle[\mathrm{e}^{\mathrm{i}/\hbar L_{\mathrm{o}}t}(1-P)L_{\mathrm{int}}b_{\mathbf{k}\beta}, A(0)]\rangle_{\mathrm{o}}, \tag{A.9}$$

in order to take into account the real and imaginary parts.

## Appendix B. Magnon mass operator for RCF

Using the Wick theorem to (A.6), one finds the contributions of first order in the $\mathcal{H}_4$ interaction part and of second order in the $\mathcal{H}_3$ and $\mathcal{H}_4$ interaction parts to the diagonal components of the mass operator:

$$\begin{aligned}\Sigma_1^{4(1)}(\mathbf{k},\omega) = {}& \frac{2}{\hbar}\sum_{\mathbf{q}}[C_{\mathrm{I}}(\mathbf{qkqk}) + C_{\mathrm{I}}(\mathbf{kqkq}) + C_{\mathrm{I}}(\mathbf{qqkk}) + C_{\mathrm{I}}(\mathbf{kkqq}) + 2C_{\mathrm{I}}(\mathbf{kqqk})\\ &+ 2C_{\mathrm{I}}(\mathbf{qkkq}) + C_{\mathrm{II}}(\mathbf{qkqk}) + C_{\mathrm{II}}(\mathbf{kqkq}) + C_{\mathrm{II}}(\mathbf{qqkk}) + C_{\mathrm{II}}(\mathbf{kkqq})]n_{\perp\mathbf{q}}\\ &+ \frac{4}{\hbar}\sum_{\mathbf{q}}E(\mathbf{qqkk})n_{\|\mathbf{q}},\end{aligned} \tag{B.1}$$

$$\begin{aligned}\Sigma_1^{3(2)}(\mathbf{k},\omega) = {}& \frac{1}{\hbar}\sum_{\mathbf{12}}\{[-(\omega+\omega_{\|\mathbf{1}}+\omega_{\perp\mathbf{2}})^{-1} + \mathrm{i}\pi\delta(\omega+\omega_{\|\mathbf{1}}+\omega_{\perp\mathbf{2}})]\Delta(\mathbf{1}+\mathbf{2}+\mathbf{k})\\ &\times [A_{\mathrm{III}}(\mathbf{12k}) + A_{\mathrm{III}}(\mathbf{1k2})]^2 + [(\omega-\omega_{\|\mathbf{1}}-\omega_{\perp\mathbf{2}})^{-1} - \mathrm{i}\pi\delta(\omega-\omega_{\|\mathbf{1}}-\omega_{\perp\mathbf{2}})]\\ &\times \Delta(\mathbf{1}+\mathbf{2}-\mathbf{k})[A_{\mathrm{II}}(\mathbf{12k}) + A_{\mathrm{IV}}(\mathbf{1k2})]^2\}(n_{\|\mathbf{1}}+n_{\perp\mathbf{2}}+1)\\ &+ \frac{1}{\hbar}\sum_{\mathbf{12}}\{[-(\omega-\omega_{\|\mathbf{1}}+\omega_{\perp\mathbf{2}})^{-1} + \mathrm{i}\pi\delta(\omega-\omega_{\|\mathbf{1}}+\omega_{\perp\mathbf{2}})]\Delta(\mathbf{1}-\mathbf{2}-\mathbf{k})\\ &\times [A_{\mathrm{I}}(\mathbf{12k}) + A_{\mathrm{I}}(\mathbf{1k2})]^2 + [(\omega+\omega_{\|\mathbf{1}}-\omega_{\perp\mathbf{2}})^{-1} - \mathrm{i}\pi\delta(\omega_{\perp\mathbf{k}}+\omega_{\|\mathbf{1}}-\omega_{\perp\mathbf{2}})]\\ &\times \Delta(\mathbf{1}-\mathbf{2}+\mathbf{k})[A_{\mathrm{II}}(\mathbf{1k2}) + A_{\mathrm{IV}}(\mathbf{12k})]^2\}(-n_{\perp\mathbf{2}}+n_{\|\mathbf{1}}),\end{aligned} \tag{B.2}$$

$$\begin{aligned}\Sigma_2^{3(2)}(\mathbf{k},\omega) = {}& \frac{1}{\hbar}\sum_{\mathbf{12}}\{[-(\omega+\omega_{\perp\mathbf{1}}+\omega_{\perp\mathbf{2}})^{-1} + \mathrm{i}\pi\delta(\omega+\omega_{\perp\mathbf{1}}+\omega_{\perp\mathbf{2}})]\Delta(\mathbf{1}+\mathbf{2}+\mathbf{k})\\ &\times A_{\mathrm{III}}(\mathbf{k12})[A_{\mathrm{III}}(\mathbf{k12}) + A_{\mathrm{III}}(\mathbf{k21})] + [(\omega-\omega_{\perp\mathbf{1}}-\omega_{\perp\mathbf{2}})^{-1}\\ &- \mathrm{i}\pi\delta(\omega-\omega_{\perp\mathbf{1}}-\omega_{\perp\mathbf{2}})]\\ &\times \Delta(\mathbf{1}+\mathbf{2}-\mathbf{k})A_{\mathrm{I}}(\mathbf{k12})[A_{\mathrm{I}}(\mathbf{k12}) + A_{\mathrm{I}}(\mathbf{k21})]\}(n_{\perp\mathbf{1}}+n_{\perp\mathbf{2}}+1)\\ &+ \frac{1}{\hbar}\sum_{\mathbf{12}}\{[-(\omega-\omega_{\perp\mathbf{1}}+\omega_{\perp\mathbf{2}})^{-1} + \mathrm{i}\pi\delta(\omega-\omega_{\perp\mathbf{1}}+\omega_{\perp\mathbf{2}})]\\ &\times \Delta(\mathbf{1}-\mathbf{2}-\mathbf{k})A_{\mathrm{IV}}(\mathbf{k12})\\ &\times [A_{\mathrm{II}}(\mathbf{k21}) + A_{\mathrm{IV}}(\mathbf{k12})] + [(\omega+\omega_{\perp\mathbf{1}}-\omega_{\perp\mathbf{2}})^{-1} - \mathrm{i}\pi\delta(\omega+\omega_{\perp\mathbf{1}}-\omega_{\perp\mathbf{2}})]\\ &\times \Delta(\mathbf{1}-\mathbf{2}+\mathbf{k})A_{\mathrm{II}}(\mathbf{k12})[A_{\mathrm{II}}(\mathbf{k12}) + A_{\mathrm{IV}}(\mathbf{k21})]\}(-n_{\perp\mathbf{2}}+n_{\perp\mathbf{1}})\\ &+ \frac{1}{\hbar}\sum_{\mathbf{12}}\{[-(\omega+\omega_{\|\mathbf{1}}+\omega_{\|\mathbf{2}})^{-1} + \mathrm{i}\pi\delta(\omega+\omega_{\|\mathbf{1}}+\omega_{\|\mathbf{2}})]\\ &\times [B_{\mathrm{II}}(\mathbf{12k})]^2 18\Delta(\mathbf{1}+\mathbf{2}+\mathbf{k})\\ &+ [(\omega-\omega_{\|\mathbf{1}}-\omega_{\|\mathbf{2}})^{-1} - \mathrm{i}\pi\delta(\omega-\omega_{\|\mathbf{1}}-\omega_{\|\mathbf{2}})]\Delta(\mathbf{1}+\mathbf{2}-\mathbf{k})[B_{\mathrm{I}}(\mathbf{k12})]^2 18\}\end{aligned}$$

$$\times\,(n_{\|\mathbf{1}}+n_{\|\mathbf{2}}+1)+\frac{1}{\hbar}\sum_{\mathbf{12}}[(\omega-\omega_{\|\mathbf{1}}+\omega_{\|\mathbf{2}})^{-1}-\mathrm{i}\pi\delta(\omega-\omega_{\|\mathbf{1}}+\omega_{\|\mathbf{2}})]$$
$$\times\,\Delta(\mathbf{1}-\mathbf{2}-\mathbf{k})[B_{\mathrm{I}}(\mathbf{12k})]^2 36(n_{\|\mathbf{2}}-n_{\|\mathbf{1}}), \tag{B.3}$$

$$\mathrm{Im}\,\Sigma_1^{4(2)}(\mathbf{k},\omega)=\frac{\pi}{2\hbar^2}\sum_{\mathbf{123}}[M_\perp(\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{k})]^2[-n_{\perp\mathbf{1}}(n_{\perp\mathbf{2}}+1)(n_{\perp\mathbf{3}}+1)$$
$$+\,(n_{\perp\mathbf{1}}+1)n_{\perp\mathbf{2}}n_{\perp\mathbf{3}}]\delta(\omega+\omega_{\perp\mathbf{1}}-\omega_{\perp\mathbf{2}}-\omega_{\perp\mathbf{3}})\Delta(\mathbf{1}-\mathbf{2}-\mathbf{3}+\mathbf{k}). \tag{B.4}$$

Here

$$\begin{aligned}M_\perp(\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{k})={}&C_{\mathrm{I}}(\mathbf{1k23})+C_{\mathrm{I}}(\mathbf{1k32})+C_{\mathrm{I}}(\mathbf{k123})+C_{\mathrm{I}}(\mathbf{k132})+C_{\mathrm{I}}(\mathbf{23k1})\\&+C_{\mathrm{I}}(\mathbf{32k1})+C_{\mathrm{I}}(\mathbf{231k})+C_{\mathrm{I}}(\mathbf{321k})+C_{\mathrm{I}}(\mathbf{213k})+C_{\mathrm{I}}(\mathbf{312k})+C_{\mathrm{I}}(\mathbf{2k31})\\&+C_{\mathrm{I}}(\mathbf{3k21})+C_{\mathrm{I}}(\mathbf{12k3})+C_{\mathrm{I}}(\mathbf{13k2})+C_{\mathrm{I}}(\mathbf{k213})+C_{\mathrm{I}}(\mathbf{k312})+C_{\mathrm{II}}(\mathbf{31k2})\\&+C_{\mathrm{II}}(\mathbf{21k3})+C_{\mathrm{II}}(\mathbf{3k12})+C_{\mathrm{II}}(\mathbf{2k13})+C_{\mathrm{II}}(\mathbf{k231})+C_{\mathrm{II}}(\mathbf{k321})\\&+C_{\mathrm{II}}(\mathbf{123k})+C_{\mathrm{II}}(\mathbf{132k}).\end{aligned} \tag{B.5}$$

The change of $C_\mu(\mathbf{1234})$ into $D_\mu(\mathbf{1234})$ in (B.5) and in (B.1) together with the changes $n_{\perp\mathbf{q}}\to n_{\|\mathbf{q}}$ and $\omega_{\perp\mathbf{k}}\to\omega_{\|\mathbf{k}}$ in (B.1) and (B.4) lead to the expressions for $\Sigma_2^{4(1)}(\mathbf{k},\omega)$ and $\mathrm{Im}\,\Sigma_2^{4(2)}(\mathbf{k},\omega)$, respectively. We do not include contributions from the part of $\mathcal{H}_4$ containing products of the different mode creation (annihilation) operators $a_{\mathbf{k}}^{(\dagger)}$ and $b_{\mathbf{k}}^{(\dagger)}$ since they are negligible on the mass surfaces defined by the equations

$$\begin{aligned}&\omega_{\perp\mathbf{k}}+\omega_{\|\mathbf{1}}-\omega_{\perp\mathbf{2}}-\omega_{\|\mathbf{3}}=0,\\&\omega_{\|\mathbf{k}}+\omega_{\perp\mathbf{1}}-\omega_{\|\mathbf{2}}-\omega_{\perp\mathbf{3}}=0.\end{aligned} \tag{B.6}$$

We omit a presentation of the detailed form of the coefficients of decomposition (A.7) of the one-particle Green functions $\mathbf{\Lambda}_A(\mathbf{k},\omega)$ useful for calculations of the dynamical susceptibility.

## Appendix C. Magnon mass operator for SSG

From (A.6), the contributions of the second order in the $\mathcal{H}_3$ and $\mathcal{H}_4$ interaction parts to the spin-wave damping coefficients of the low-energy shear mode take the form

$$\Gamma_{\mathbf{k}-}^{3(2)}=\frac{\pi}{\hbar^2}\sum_{\mathbf{12}}[A_{--}(\mathbf{12k})+A_{--}(\mathbf{1k2})]^2\Delta(\mathbf{1}+\mathbf{2}-\mathbf{k})\delta(\omega_{\mathbf{k}-}-\omega_{\|\mathbf{1}}-\omega_{\mathbf{2}-})$$
$$\times\,(n_{\|\mathbf{1}}+n_{\mathbf{2}-}+1)+\frac{\pi}{\hbar^2}\sum_{\mathbf{12}}[A_{--}(\mathbf{1k2})+A_{--}(\mathbf{12k})]^2$$
$$\times\,\Delta(\mathbf{1}-\mathbf{2}+\mathbf{k})\delta(\omega_{\mathbf{k}-}+\omega_{\|\mathbf{1}}-\omega_{\mathbf{2}-})(-n_{\mathbf{2}-}+n_{\|\mathbf{1}}), \tag{C.1}$$

$$\Gamma_{\mathbf{k}-}^{4(2)}=\frac{\pi}{2\hbar^2}\sum_{\mathbf{123}}[M_-(\mathbf{123k})]^2[n_{\mathbf{1}-}(n_{\mathbf{2}-}+1)(n_{\mathbf{3}-}+1)-(n_{\mathbf{1}-}+1)n_{\mathbf{2}-}n_{\mathbf{3}-}]$$
$$\times\,\Delta(\mathbf{1}-\mathbf{2}-\mathbf{3}+\mathbf{k})\delta(\omega_{\mathbf{k}-}+\omega_{\mathbf{1}-}-\omega_{\mathbf{2}-}-\omega_{\mathbf{3}-}), \tag{C.2}$$

where

$$\begin{aligned}M_-(\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{k})={}&C(\mathbf{213k})+C(\mathbf{312k})+C(\mathbf{2k31})+C(\mathbf{3k21})+C(\mathbf{12k3})+C(\mathbf{13k2})\\&+C(\mathbf{k213})+C(\mathbf{k312})+C(\mathbf{31k2})+C(\mathbf{21k3})+C(\mathbf{3k12})\\&+C(\mathbf{2k13})+C(\mathbf{k231})+C(\mathbf{k321})+C(\mathbf{123k})+C(\mathbf{132k}).\end{aligned} \tag{C.3}$$

We do not include contributions from the part of $\mathcal{H}_4$ containing products of the different mode creation (annihilation) operators $a_{\mathbf{k}-}^{(\dagger)}$ and $b_{\mathbf{k}}^{(\dagger)}$, since they are negligible on the relevant mass surfaces defined by equations (B.6) with the change $\omega_{\perp\mathbf{k}}\to\omega_{\mathbf{k}-}$.

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